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Coherent states for the harmonic series irreducible representations of U(p, q): I. Unitary operator coherent states

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Abstract. Unitary operator coherent states, as defined by Klauder, Perelomov and Gilmore, are considered for the harmonic series irreducible representations of U(p, q). Various properties of these states are reviewed, including their reproducing kernel and measure explicit forms, as well as the representation they lead to for the U(p, q) generators. Starting from a contraction of the su(p, q) Lie algebra, both Dyson and Holstein-Primakoff boson realisations of u(p, q) are obtained in terms of pq pairs of boson creation, annihilation operators and of the generators of a U(p) × U(q) intrinsic group. Such boson realisations are applied to determine the matrix elements of the U(p, q) generators between discrete bases classified according to the chain U(p, q) ⊃ U(p) × U(q). Finally, the isomorphism between so(4, 2) and su(2, 2) is employed to derive the corresponding properties of the SO(4, 2) unitary operator coherent states for application in atomic physics.

1. Introduction

It is well known that the harmonic oscillator coherent states (CS) (Schrödinger 1926), also referred to in the literature as Glauber's standard CS (Glauber 1963) or the CS associated with the Heisenberg-Weyl group N(1), can be defined in several equivalent ways (for a recent review on the standard CS and their extensions see Gilmore and Feng (1983)). In particular, they are both unitary operator and annihilation operator CS, since they can be obtained either by applying a unitary transformation to the harmonic oscillator ground state or as the eigenstates of the harmonic oscillator annihilation operator corresponding to a complex eigenvalue.

In extending the notion of CS to other physical systems, the various CS definitions lose their equivalency, thereby giving rise to different sets of generalised CS. Both definitions of the standard CS as unitary operator or annihilation operator CS admit of group theoretical extensions, wherein the CS are associated with a dynamical Lie group of the considered physical system. Such generalisations were proposed by Klauder (1963, 1964), Perelomov (1972, 1977) and Gilmore (1972, 1974a), and by Barut and Girardello (1971). The former is valid for any Lie group, either compact or non-compact, whereas the latter can only be applied to non-compact groups.

Apart from SO(2, 1) and its locally isomorphic groups SU(1, 1), Sl(2, R) and Sp(2, R), as considered in the earliest works (Barut and Girardello 1971, Perelomov

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1972, 1977, Gilmore 1974a), most detailed results were obtained for compact groups (Dobaczewski 1981, 1982 and references quoted therein, Hecht and Elliott 1985, Hecht 1985). Only recently, generalised CS were constructed for a class of discrete series irreducible representations (irreps) of Sp(2d, R), namely the harmonic series irreps, frequently encountered in physical applications. They include both unitary operator CS (Kramer 1982, Deenen and Quesne 1984a, Rowe 1984, Kramer and Papadopolos 1986, Kramer *et al* 1986, Quesne 1986a), and annihilation operator ones (Deenen and Quesne 1984a, Quesne 1986b), as well as a further generalisation of CS, using both continuous and discrete labels and termed either partially CS (Deenen and Quesne 1984b) or vector CS (Rowe *et al* 1985). In addition, the relations between CS representations and boson realisations of the sp(2d, R) algebra have been extensively studied (Mlodinow and Papanicolaou 1981, Deenen and Quesne 1982, 1983, 1984a, b, 1985, Rowe 1984, Castaños *et al* 1985a, b).

The aim of the present series of two papers is to present detailed results for the CS of the pseudo-unitary groups U(p, q), corresponding to the harmonic series irreps (Kashiwara and Vergne 1978, King and Wybourne 1985, Quesne 1986c). The latter can be labelled by $[\boldsymbol{k}; \boldsymbol{k}']$, where $[\boldsymbol{k}] = [k_1 \ldots k_p]$ and $[\boldsymbol{k}'] = [k'_1 \ldots k'_q]$ are two partitions and they play an important role both in applications of group theory to atomic physics (Quesne 1986c) and in SU(n) representation theory (Deenen and Quesne 1986, Quesne 1987). In the latter case, their relevance follows from the complementarity relationship between U(n) and U(p, q) for mixed U(n) irreps (Kashiwara and Vergne 1978, King and Wybourne 1985, Quesne 1986c, d), while in the former it results from the isomorphism of the Lie algebras so(4, 2) and su(2, 2) and the property that SO(4, 2) is a dynamical group for the hydrogen atom (Barut and Kleinert 1967) and for two-electron atoms (Wulfman and Kumei 1973, Wulfman 1973). The present series of papers will, in particular, attempt to provide solutions to the long-standing problem of finding the counterpart for the hydrogen atom of the harmonic oscillator CS. Such solutions based upon SO(4, 2) differ from other available ones (Nieto 1980, Gerry 1986 and references quoted therein). Note, however, that Mostowski (1977) has already published some partial results on the same type of CS as those considered in the present paper.

While paper II will be devoted to the annihilation operator CS, the present paper deals with the unitary operator CS. In § 2, the U(p, q) harmonic series $[\boldsymbol{k}; \boldsymbol{k}']$ are reviewed. In § 3, the unitary operator CS are defined and their properties listed in the case where $k_1 = \ldots = k_p = k$ and $k'_1 = \ldots = k'_q = k'$. In § 4, these results are extended to the irreps for which $k_1, \ldots, k_p$ and $k'_1, \ldots, k'_q$ are not all equal, with special emphasis on the U(2, 1) and U(2, 2) cases. In § 5, a contraction of su(p, q) leading to a boson algebra is considered. In § 6, both Dyson (1956) and Holstein and Primakoff (1940) boson realisations of u(p, q) are obtained. In § 7, they are used to determine the matrix elements of the generators between discrete bases. Finally, in § 8, the isomorphism between su(2, 2) and so(4, 2) is applied to obtain SO(4, 2) unitary operator CS.

2. Harmonic series irreducible representations of U(p, q)

The U(p, q) generators can be realised in terms of dn boson creation operators η_{is}, $i = 1, \ldots, d$, $s = 1, \ldots, n$, and their corresponding annihilation operators $\xi_{is} = (\eta_{is})^\dagger$, as follows (Quesne 1986c):

$$P_{ij} = \begin{cases} E_{ij} & \text{if } i, j = 1, \ldots, p \\ E_{ji} & \text{if } i, j = p+1, \ldots, d \\ D^\dagger_{ij} & \text{if } i = 1, \ldots, p \text{ and } j = p+1, \ldots, d \\ D_{ij} & \text{if } i = p+1, \ldots, d \text{ and } j = 1, \ldots, p \end{cases} \tag{2.1}$$

where we assume $p \geqslant q$, and define $d = p + q$, and

$$D^\dagger_{ij} = \sum_{s=1}^{n} \eta_{is}\eta_{js} \qquad D_{ij} = \sum_{s=1}^{n} \xi_{is}\xi_{js}$$
$$E_{ij} = \tfrac{1}{2} \sum_{s=1}^{n} (\eta_{is}\xi_{js} + \xi_{js}\eta_{is}) = \sum_{s=1}^{n} \eta_{is}\xi_{js} + \tfrac{1}{2} n\, \delta_{ij}. \tag{2.2}$$

They satisfy the hermiticity properties

$$(P_{ij})^\dagger = P_{ji} \tag{2.3}$$

and the commutation relations

$$[P_{ij}, P_{kl}] = g_{jk} P_{il} - g_{li} P_{kj} \tag{2.4}$$

where $g_{ij} = \varepsilon_i \delta_{ij}$ and $\varepsilon_i = +1$ or -1 according as $i = 1, \ldots, p$ or $i = p+1, \ldots, d$. For future convenience, we shall henceforth denote the d values of index i by α and $p + \beta$, where α and β go from 1 to p and 1 to q respectively. With this convention, the operators of (2.1) become $E_{\alpha\alpha'}$, $E_{p+\beta,p+\beta'}$, $D^\dagger_{\alpha,p+\beta}$ and $D_{\alpha,p+\beta}$.

From (2.3) and (2.4), the operators $E_{\alpha\alpha'}$ and $E_{p+\beta,p+\beta'}$ generate the maximal compact subgroup $U(p) \times U(q)$ of $U(p, q)$. The set of generators (2.1) can be divided into three subsets of raising, weight and lowering type as follows:

$$D^\dagger_{\alpha,p+\beta},\ E_{\alpha\alpha'}(\alpha < \alpha'),\ E_{p+\beta,p+\beta'}(\beta < \beta'); \qquad E_{\alpha\alpha},\ E_{p+\beta,p+\beta};$$
$$D_{\alpha,p+\beta},\ E_{\alpha\alpha'}(\alpha > \alpha'),\ E_{p+\beta,p+\beta'}(\beta > \beta') \tag{2.5}$$

where the subsets are separated by semicolons.

In the realisation (2.1) and (2.2), $U(p, q)$ has only positive discrete series irreps, the so-called harmonic series ones, characterised by their lowest weight $\{k_p + \frac{1}{2}n, \ldots, k_1 + \frac{1}{2}n;\ k'_q + \frac{1}{2}n, \ldots, k'_1 + \frac{1}{2}n\}$, that we denote in short by $[\boldsymbol{k}; \boldsymbol{k}']$ (Quesne 1986c). Here $[\boldsymbol{k}] = [k_1 \ldots k_p]$ and $[\boldsymbol{k}'] = [k'_1 \ldots k'_q]$ are two partitions. The irreps for which $k_1, \ldots, k_p, k'_1, \ldots, k'_q$ do not vanish can be realised only provided that $n \geqslant d$. In the remaining cases, the following conditions have to be fulfilled:

$$\begin{aligned}
&k_{n-q+\sigma+1} = \ldots = k_p = k'_{q-\sigma+1} = \ldots = k'_q = 0 && \text{for some } \sigma \in \{0, 1, \ldots, d-n\} \\
& && \text{if} \quad p \leqslant n < d \\
&k_{n-q+\sigma+1} = \ldots = k_p = k'_{q-\sigma+1} = \ldots = k'_q = 0 && \text{for some } \sigma \in \{0, 1, \ldots, q\} \\
& && \text{if} \quad q \leqslant n < p \\
&k_{\sigma+1} = \ldots = k_p = k'_{n-\sigma+1} = \ldots = k'_q = 0 && \text{for some } \sigma \in \{0, 1, \ldots, n\} \\
& && \text{if} \quad n < q.
\end{aligned} \tag{2.6}$$

The lowest-weight state (LWS) of an irrep $[\boldsymbol{k};\ \boldsymbol{k}']$ satisfies the following equations:

$$\begin{aligned}
&D_{\alpha,p+\beta}|\mathrm{LWS}\rangle = 0\\
&E_{\alpha\alpha'}|\mathrm{LWS}\rangle = E_{p+\beta,p+\beta'}|\mathrm{LWS}\rangle = 0 \qquad \alpha>\alpha',\ \beta>\beta'\\
&E_{\alpha\alpha}|\mathrm{LWS}\rangle = (k_{p+1-\alpha}+n/2)|\mathrm{LWS}\rangle\\
&E_{p+\beta,p+\beta}|\mathrm{LWS}\rangle = (k'_{q+1-\beta}+n/2)|\mathrm{LWS}\rangle
\end{aligned} \tag{2.7}$$

and is also the LWS of a $\mathrm{U}(p)\times\mathrm{U}(q)$ irrep characterised by $[\boldsymbol{k}]\times[\boldsymbol{k}']$. A solution of (2.7) is given by (Quesne 1986c, 1987)

$$\begin{aligned}
|\mathrm{LWS}\rangle = &\left[\left(\prod_{\alpha<\alpha'}^{p}(k_\alpha-k_{\alpha'}+\alpha'-\alpha)\right)\left(\prod_{\alpha=1}^{p}(k_\alpha+p-\alpha)!\right)^{-1}\right]^{1/2}\\
&\times\left[\left(\prod_{\beta<\beta'}^{q}(k'_\beta-k'_{\beta'}+\beta'-\beta)\right)\left(\prod_{\beta=1}^{q}(k'_\beta+q-\beta)!\right)^{-1}\right]^{1/2}\\
&\times\left(\prod_{\alpha=1}^{p}(\eta_{p-\alpha+1\ldots p,1\ldots\alpha})^{k_\alpha-k_{\alpha+1}}\right)\left(\prod_{\beta=1}^{q}(\eta_{d-\beta+1\ldots d,n-\beta+1\ldots n})^{k'_\beta-k'_{\beta+1}}\right)|0\rangle
\end{aligned} \tag{2.8}$$

where $k_{p+1}=k'_{q+1}=0$ and $\eta_{i_1\ldots i_r,s_1\ldots s_r}$ denotes the determinant of order r formed from the rows $i_1,\ldots,i_r$ and the columns $s_1,\ldots,s_r$ of the $d\times n$ matrix $\|\eta_{is}\|$.

From the LWS (2.8), bases of the $\mathrm{U}(p)\times\mathrm{U}(q)$ irrep $[\boldsymbol{k}]\times[\boldsymbol{k}']$ are generated by applying polynomials in $E_{\alpha\alpha'}$ and $E_{p+\beta,p+\beta'}$. The resulting states $|(k),(k')\rangle$ can be characterised by $\mathrm{U}(p)$ and $\mathrm{U}(q)$ Gel'fand patterns, denoted by (k) and (k'), respectively. The remaining bases of the representation space $\mathscr{F}_{[\boldsymbol{k};\boldsymbol{k}']}$ of $[\boldsymbol{k};\ \boldsymbol{k}']$ are obtained by applying polynomials in $D^{\dagger}_{\alpha,p+\beta}$ to the states $|(k),(k')\rangle$. Such polynomials $P_{[\boldsymbol{h}^s](h^s)(h^{s'})}(D^{\dagger}_{\alpha,p+\beta})$ can be specified by a $\mathrm{U}(p)\times\mathrm{U}(q)$ irrep $[\boldsymbol{h}^s\dot{0}]\times[\boldsymbol{h}^s]$ and by the corresponding Gel'fand patterns (h^s) and $(h^{s'})$. Here the $\mathrm{U}(p)$ and $\mathrm{U}(q)$ irreps are characterised by the same partition into q integers $h_1^s,\ldots,h_q^s$ and a dot over a numeral means that this numeral is repeated as often as necessary. Bases of $\mathscr{F}_{[\boldsymbol{k};\boldsymbol{k}']}$, classified according to the chain $\mathrm{U}(p,q)\supset\mathrm{U}(p)\times\mathrm{U}(q)$, are therefore given by (Quesne 1987)

$$\begin{aligned}
|\omega[\boldsymbol{h}][\boldsymbol{h}'](h)(h')\rangle = B_\omega &\sum_{(k)(k')(h^s)(h^{s'})}\left\langle\begin{matrix}[\boldsymbol{k}] & [\boldsymbol{h}^s\dot{0}]\\ (k)' & (h^s)\end{matrix}\middle|\chi;\begin{matrix}[\boldsymbol{h}]\\ (h)\end{matrix}\right\rangle\\
&\times\left\langle\begin{matrix}[\boldsymbol{k}'] & [\boldsymbol{h}^s]\\ (k')' & (h^{s'})\end{matrix}\middle|\chi';\begin{matrix}[\boldsymbol{h}']\\ (h')\end{matrix}\right\rangle P_{[\boldsymbol{h}^s](h^s)(h^{s'})}(D^{\dagger}_{\alpha,p+\beta})|(k),(k')\rangle
\end{aligned} \tag{2.9}$$

where $\langle\ ,\ |\ \rangle$ denotes an $\mathrm{SU}(p)$ or $\mathrm{SU}(q)$ Wigner coefficient, χ and χ' are multiplicity labels, ω is shorthand for $[\boldsymbol{h}^s]\chi\chi'$ and B_ω is some normalisation coefficient whose dependence on $[\boldsymbol{k}]$, $[\boldsymbol{k}']$, $[\boldsymbol{h}]$, $[\boldsymbol{h}']$ has not been indicated. With the choice made for B_ω in § 5, the states (2.9) are not normalised to unity, nor are they orthogonal with respect to $[\boldsymbol{h}^s]$, which cannot be associated directly with eigenvalues of Hermitian operators.

The $\mathrm{SU}(p,q)$ subgroup of $\mathrm{U}(p,q)$ is generated by the operators

$$\bar{P}_{ij}=P_{ij}-(1/d)g_{ij}G_1 \qquad i,j=1,\ldots,d \tag{2.10}$$

where

$$G_1=\sum_i \varepsilon_i P_{ii}=\sum_\alpha E_{\alpha\alpha}-\sum_\beta E_{p+\beta,p+\beta} \tag{2.11}$$

is the first-order Casimir operator of U(p, q). An alternative choice for the SU(p, q) generators consists of $D^{\dagger}_{\alpha,p+\beta}$, $D_{\alpha,p+\beta}$ and the operators

$$\begin{aligned}
\bar{E}_{\alpha\alpha'} &= E_{\alpha\alpha'} - (1/p)\,\delta_{\alpha\alpha'} \sum_{\alpha''} E_{\alpha''\alpha''} \\
\bar{E}_{p+\beta,p+\beta'} &= E_{p+\beta,p+\beta'} - (1/q)\,\delta_{\beta\beta'} \sum_{\beta''} E_{p+\beta'',p+\beta''} \\
\bar{E} &= \sum_i \left(E_{ii} - \frac{\varepsilon_i}{d} G_1 \right) = \frac{2}{d}\left(q \sum_\alpha E_{\alpha\alpha} + p \sum_\beta E_{p+\beta,p+\beta} \right)
\end{aligned} \tag{2.12}$$

generating the maximal compact subgroup SU(p)×SU(q)×U(1) of SU(p, q). Under restriction of U(p, q) to SU(p, q), the irreps $[\boldsymbol{k}; \boldsymbol{k}']$ remain irreducible and satisfy the equivalence relation

$$[k_1 + c, \ldots, k_p + c; k'_1 - c, \ldots, k'_q - c] \sim [k_1 \ldots k_p; k'_1 \ldots k'_q] \tag{2.13}$$

for c any integer subject to the conditions $k_\alpha + c \geq 0$ and $k'_\beta - c \geq 0$.

Having reviewed the U(p, q) harmonic series irreps $[\boldsymbol{k}; \boldsymbol{k}']$ and their discrete bases, we can now proceed to define and study the corresponding unitary operator CS. In § 3, we shall carry on this programme for a special type of U(p, q) irreps, leaving the discussion of the general case until § 4.

3. Coherent states for the irreducible representations $[\dot{k}; \dot{k}']$

Let us consider the class of U(p, q) irreps for which $k_1 = \ldots = k_p = k$ and $k'_1 = \ldots = k'_q = k'$ and denote them by $[\dot{k}; \dot{k}']$. They are the analogue of the Sp(2d, R) irreps $\langle(\lambda + \frac{1}{2}n)^d\rangle$, whose unitary operator CS were studied by Kramer (1982) and Deenen and Quesne (1984a). The LWS of a $[\dot{k}; \dot{k}']$ irrep belongs to a one-dimensional irrep of U(p)×U(q), and from (2.8) it becomes

$$|\text{LWS}\rangle = \left[\left(\prod_{\alpha=1}^{p} (p-\alpha)!/(k+p-\alpha)! \right) \left(\prod_{\beta=1}^{q} (q-\beta)!/(k'+q-\beta)! \right) \right]^{1/2} \times (\eta_{1\ldots p,1\ldots p})^k (\eta_{p+1\ldots d,n-q+1\ldots n})^{k'} |0\rangle. \tag{3.1}$$

Following Klauder (1963, 1964), Perelomov (1972, 1977) and Gilmore (1972, 1974a), the system of unitary operator CS corresponding to $[\dot{k}; \dot{k}']$ and the reference state |LWS⟩ is obtained by applying to the latter the unitary operator representing an element g of U(p, q) in $\mathscr{F}_{[\dot{k};\dot{k}']}$ and by letting g run over the whole group. Since the U(p, q) generators, with the exception of $D^{\dagger}_{\alpha,p+\beta}$, have a trivial effect on (3.1), giving either zero or the same state multiplied by a constant, the U(p, q) CS can be written as

$$|\boldsymbol{u}\rangle = \exp\left(\sum_{\alpha\beta} u^*_{\alpha\beta} D^{\dagger}_{\alpha,p+\beta} \right) |\text{LWS}\rangle \tag{3.2}$$

where * denotes complex conjugation and the summation runs over α and β from 1 to p and 1 to q, respectively. The parameters $u_{\alpha\beta}$ specify the points of the coset space U(p, q)/[U(p)×U(q)], where U(p)×U(q) is the stability group of the reference state (3.1). Hence they form a complex $p \times q$ matrix $\boldsymbol{u}$, subject to the condition that $\boldsymbol{I} - \boldsymbol{u}\boldsymbol{u}^{\dagger}$ (or equivalently $\boldsymbol{I} - \boldsymbol{u}^{\dagger}\boldsymbol{u}$) be a positive-definite (Hermitian) matrix (Hua 1963, Gilmore 1974b).

The CS $|\boldsymbol{u}\rangle$ form a non-orthogonal family of states, their overlap being given by

$$\hat{K}(\boldsymbol{u}'; \boldsymbol{u}^*) = \langle \boldsymbol{u}' | \boldsymbol{u} \rangle = (\det \boldsymbol{U})^{-k-k'-n} = (\det \boldsymbol{V})^{-k-k'-n} \tag{3.3}$$

where

$$\boldsymbol{U} = \boldsymbol{I} - \boldsymbol{u}'\boldsymbol{u}^\dagger \qquad \boldsymbol{V} = \boldsymbol{I} - \boldsymbol{u}^\dagger \boldsymbol{u}'. \tag{3.4}$$

The proof of (3.3) goes in three steps. First, the relation

$$\begin{aligned}\exp\left(\sum_{\alpha\beta} u'_{\alpha\beta} D_{\alpha,p+\beta}\right) \exp\left(\sum_{\alpha\beta} u^*_{\alpha\beta} D^\dagger_{\alpha,p+\beta}\right) \\ = \exp\left(\sum_{\alpha\beta} a_{\alpha\beta} D^\dagger_{\alpha,p+\beta}\right) \exp\left(\sum_{\alpha\alpha'} b_{\alpha\alpha'} E_{\alpha\alpha'} + \sum_{\beta\beta'} c_{\beta\beta'} E_{p+\beta,p+\beta'}\right) \\ \times \exp\left(\sum_{\alpha\beta} d_{\alpha\beta} D_{\alpha,p+\beta}\right)\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}&\boldsymbol{a} = \tilde{\boldsymbol{U}}^{-1}\boldsymbol{u}^* = \boldsymbol{u}^*\tilde{\boldsymbol{V}}^{-1} \qquad \exp \boldsymbol{b} = \tilde{\boldsymbol{U}}^{-1} \\ &\exp \boldsymbol{c} = \boldsymbol{V}^{-1} \qquad \boldsymbol{d} = \boldsymbol{U}^{-1}\boldsymbol{u}' = \boldsymbol{u}'\boldsymbol{V}^{-1}\end{aligned} \tag{3.6}$$

and $\tilde{}$ denotes transposition, is established by using a matrix representation of the complex extension $\mathrm{Gl}(p+q, C)$ of $\mathrm{U}(p, q)$ (Gilmore 1974c, Deenen and Quesne 1984b). Second, the property

$$\begin{aligned}\langle \text{LWS} | \exp\left(\sum_{\alpha\alpha'} b_{\alpha\alpha'} E_{\alpha\alpha'} + \sum_{\beta\beta'} c_{\beta\beta'} E_{p+\beta,p+\beta'}\right) | \text{LWS} \rangle \\ = \mathscr{D}^{[k]}(\tilde{\boldsymbol{U}}^{-1}) \mathscr{D}^{[k']}(\boldsymbol{V}^{-1})\end{aligned} \tag{3.7}$$

is taken into account. In (3.7), $\mathscr{D}^{[k]}$ and $\mathscr{D}^{[k']}$ are (one-dimensional) representation matrices of $\mathrm{Gl}(p, C)$ and $\mathrm{Gl}(q, C)$, respectively. Finally, a theorem of Brunet and Seligman (1975) for the representation matrices of $\mathrm{Gl}(n, C)$ is applied, leading to (3.3).

Provided that k and k' fulfil the condition $k + k' + n > d$, the CS satisfy a unity resolution relation:

$$\int \mathrm{d}\hat{\sigma}(\boldsymbol{u}) |\boldsymbol{u}\rangle\langle \boldsymbol{u}| = I_{[\dot{k};\dot{k}']} \tag{3.8}$$

with the representation space $\mathscr{F}_{[\dot{k};\dot{k}']}$. In (3.8), the integration is carried out over the origin-centred unit ball and the measure $\mathrm{d}\hat{\sigma}(\boldsymbol{u})$ is given by

$$\mathrm{d}\hat{\sigma}(\boldsymbol{u}) = \hat{f}(\boldsymbol{u}, \boldsymbol{u}^*)\, \mathrm{d}\boldsymbol{u}\, \mathrm{d}\boldsymbol{u}^* \tag{3.9}$$

where

$$\hat{f}(\boldsymbol{u}, \boldsymbol{u}^*) = \hat{A}\, (\det \boldsymbol{U})^{k+k'+n-d} \tag{3.10}$$

$\boldsymbol{U}$ is obtained from (3.4) by setting $\boldsymbol{u}' = \boldsymbol{u}$ and

$$\hat{A} = \pi^{-pq} \prod_{\beta=1}^{q} [(k+k'+n-q+\beta-1)!/(k+k'+n-d+\beta-1)!]. \tag{3.11}$$

The proof of (3.8)-(3.10) consists in showing that the operator on the left-hand side of (3.8) commutes with all the $\mathrm{U}(p, q)$ generators, thence by Schur's lemma is a multiple of $I_{[\dot{k};\dot{k}']}$. The correct normalisation (3.11) is ensured by imposing that the expectation value of the operator in the state (3.1) be equal to 1. The required integral has been calculated by Hua (1963).

Whenever the condition $k+k'+n>d$ is fulfilled, the CS $|\boldsymbol{u}\rangle$ form an overcomplete set of states with a reproducing kernel given in (3.3). Such a set can be used as a continuous basis in $\mathscr{F}_{[\dot{k};\dot{k}']}$. Any vector $|\psi\rangle$ in $\mathscr{F}_{[\dot{k};\dot{k}']}$ is then represented by an analytic function in the variables $u_{\alpha\beta}$:

$$\hat{\psi}(\boldsymbol{u})=\langle\boldsymbol{u}\,|\,\psi\rangle \tag{3.12}$$

and any operator X acting in $\mathscr{F}_{[\dot{k};\dot{k}']}$ by a partial differential operator $\hat{\mathscr{X}}$ with respect to $\boldsymbol{u}$:

$$\langle\boldsymbol{u}|X|\psi\rangle=\hat{\mathscr{X}}\langle\boldsymbol{u}\,|\,\psi\rangle. \tag{3.13}$$

In particular, the $\mathrm{U}(p,q)$ generators are represented by the first-order partial differential operators

$$\begin{aligned}
\hat{\mathscr{E}}_{\alpha\alpha'}&=[\boldsymbol{u}\tilde{\boldsymbol{\nabla}}+(k+\tfrac{1}{2}n)\boldsymbol{I}]_{\alpha\alpha'}\\
\hat{\mathscr{E}}_{p+\beta,p+\beta'}&=[\tilde{\boldsymbol{u}}\boldsymbol{\nabla}+(k'+\tfrac{1}{2}n)\boldsymbol{I}]_{\beta\beta'}\\
\hat{\mathscr{D}}^{\dagger}_{\alpha,p+\beta}&=\{[\boldsymbol{u}\tilde{\boldsymbol{\nabla}}+(k+k'+n-p)\boldsymbol{I}]\boldsymbol{u}\}_{\alpha\beta}\\
\mathscr{D}_{\alpha,p+\beta}&=\nabla_{\alpha\beta}
\end{aligned} \tag{3.14}$$

where $\boldsymbol{I}$ is the $p\times p$ or $q\times q$ unit matrix as the case may be and $\boldsymbol{\nabla}$ is the $p\times q$ matrix whose elements are

$$\nabla_{\alpha\beta}=\partial/\partial u_{\alpha\beta}. \tag{3.15}$$

It is straightforward to check that the operators (3.14) satisfy the $\mathrm{U}(p,q)$ commutation relations (2.4), as well as the hermiticity conditions (2.3) with respect to the measure defined in (3.9) and (3.10).

In the next section, we shall extend the definition and properties of the CS introduced in the present section to arbitrary irreps $[\boldsymbol{k};\boldsymbol{k}']$.

4. Coherent states for the irreducible representations $[\boldsymbol{k};\boldsymbol{k}']$

Whenever $k_1,\ldots,k_p$ or $k'_1,\ldots,k'_q$ are not all equal, the stability group of the LWS (2.8) is a proper subgroup $\mathrm{H}=\mathrm{H}(p)\times\mathrm{H}(q)$ of the maximal compact subgroup $\mathrm{U}(p)\times\mathrm{U}(q)$. The unitary operator CS corresponding to such an irrep and a reference state exist in one-to-one correspondence with the points of the coset space $\mathrm{U}(p,q)/\mathrm{H}$ (Klauder 1963, 1964, Perelomov 1972, 1977, Gilmore 1972, 1974a). In analogy with the CS for the $\mathrm{Sp}(2d,\boldsymbol{R})$ positive discrete series irreps (Rowe 1984, Kramer and Papadopolos 1986, Quesne 1986a), it is advantageous to adopt a parametrisation of the coset space corresponding to the factorisation

$$\mathrm{U}(p,q)/\mathrm{H}=\{\mathrm{U}(p,q)/[\mathrm{U}(p)\times\mathrm{U}(q)]\}\{[\mathrm{U}(p)\times\mathrm{U}(q)]/\mathrm{H}\} \tag{4.1}$$

where

$$[\mathrm{U}(p)\times\mathrm{U}(q)]/H=[\mathrm{U}(p)/\mathrm{H}(p)][\mathrm{U}(q)/\mathrm{H}(q)]. \tag{4.2}$$

Accordingly, the unitary operator CS for the irreps $[\boldsymbol{k};\boldsymbol{k}']$ are defined by

$$|\boldsymbol{u},\boldsymbol{y},\boldsymbol{z}\rangle=\exp\left(\sum_{\alpha\beta}u^{*}_{\alpha\beta}D^{\dagger}_{\alpha,p+\beta}\right)|\boldsymbol{y},\boldsymbol{z}\rangle \tag{4.3}$$

where $\boldsymbol{u}$ is the same matrix as in § 3 and y (z) denotes a set of parameters specifying the points of the coset space $\mathrm{U}(p)/\mathrm{H}(p)$ $[\mathrm{U}(q)/\mathrm{H}(q)]$. In other words, the states $|y, z\rangle$ are $\mathrm{U}(p)\times\mathrm{U}(q)$ CS corresponding to the irrep $[\boldsymbol{k}]\times[\boldsymbol{k}']$ and the reference state (2.8). Hence, they can be written as

$$|y, z\rangle = [M(y)N(z)]^{\dagger}|\mathrm{LWS}\rangle \tag{4.4}$$

where the commuting operators $M(y)$ and $N(z)$ depend on $E_{\alpha\alpha'}$ and $E_{p+\beta,p+\beta'}$, respectively.

The overlap of the states (4.3) can be determined in the same way as that of the states (3.2). From (3.5) and from a relation similar to (3.7), one finds

$$\begin{aligned}\hat{K}(\boldsymbol{u}', y', z'; \boldsymbol{u}^*, y^*, z^*) &= \langle \boldsymbol{u}', y', z' | \boldsymbol{u}, y, z\rangle \\ &= \mathscr{D}^{[\boldsymbol{k}]}_{(\min)(\min)}(\boldsymbol{Y}'\tilde{\boldsymbol{U}}^{-1}\boldsymbol{Y}^{\dagger})\mathscr{D}^{[\boldsymbol{k}']}_{(\min)(\min)}(\boldsymbol{Z}'\boldsymbol{V}^{-1}\boldsymbol{Z}^{\dagger}).\end{aligned} \tag{4.5}$$

Here $\mathscr{D}^{[\boldsymbol{k}]}$ and $\mathscr{D}^{[\boldsymbol{k}']}$ are the K- and K'-dimensional representation matrices of $\mathrm{Gl}(p, C)$ and $\mathrm{Gl}(q, C)$ corresponding to the irreps $[\boldsymbol{k}]$ and $[\boldsymbol{k}']$, respectively, (min) denotes the lowest-weight Gel'fand patterns of these irreps, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are the realisations of $M(y)$ and $N(z)$ by $p\times p$ and $q\times q$ matrices, respectively, and $\boldsymbol{Y}'$, $\boldsymbol{Z}'$ only differ from $\boldsymbol{Y}$, $\boldsymbol{Z}$ by the substitution of y', z' for y, z.

For most irreps $[\boldsymbol{k}; \boldsymbol{k}']$, the CS $|\boldsymbol{u}, y, z\rangle$ satisfy a unity resolution relation within the representation space $\mathscr{F}_{[\boldsymbol{k};\boldsymbol{k}']}$:

$$\int \mathrm{d}\hat{\sigma}(\boldsymbol{u}, y, z)|\boldsymbol{u}, y, z\rangle\langle \boldsymbol{u}, y, z| = I_{[\boldsymbol{k};\boldsymbol{k}']} \tag{4.6}$$

with some measure

$$\mathrm{d}\hat{\sigma}(\boldsymbol{u}, y, z) = \hat{f}(\boldsymbol{u}, \boldsymbol{u}^*, y, y^*, z, z^*)\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{u}^*\,\mathrm{d}y\,\mathrm{d}y^*\,\mathrm{d}z\,\mathrm{d}z^*. \tag{4.7}$$

The representation of the $\mathrm{U}(p, q)$ generators by first-order partial differential operators is now

$$\begin{aligned}\hat{\mathscr{E}}_{\alpha\alpha'} &= [\boldsymbol{u}\tilde{\boldsymbol{\nabla}} + \mathring{\boldsymbol{\mathscr{E}}}^{(p)}]_{\alpha\alpha'} \\ \hat{\mathscr{E}}_{p+\beta,p+\beta'} &= [\tilde{\boldsymbol{u}}\boldsymbol{\nabla} + \mathring{\boldsymbol{\mathscr{E}}}^{(q)}]_{\beta\beta'} \\ \hat{\mathscr{D}}^{\dagger}_{\alpha,p+\beta} &= [\mathring{\boldsymbol{\mathscr{E}}}^{(p)}\boldsymbol{u} + \boldsymbol{u}\tilde{\mathring{\boldsymbol{\mathscr{E}}}}^{(q)} + (\boldsymbol{u}\tilde{\boldsymbol{\nabla}} - p\boldsymbol{I})\boldsymbol{u}]_{\alpha\beta} \\ \hat{\mathscr{D}}_{\alpha,p+\beta} &= \nabla_{\alpha\beta}\end{aligned} \tag{4.8}$$

where $\mathring{\mathscr{E}}^{(p)}_{\alpha\alpha'}$ and $\mathring{\mathscr{E}}^{(q)}_{\beta\beta'}$ denote the partial differential operators representing $E_{\alpha\alpha'}$ and $E_{p+\beta,p+\beta'}$ in the $\mathrm{U}(p)$ and $\mathrm{U}(q)$ CS representations corresponding to the irreps $[\boldsymbol{k}]$ and $[\boldsymbol{k}']$, respectively.

In appendix 1, explicit expressions of the reproducing kernel, the weight function and the generator representation are given for the U(2, 1) and U(2, 2) CS.

This concludes our review of the $\mathrm{U}(p, q)$ CS properties. In the remaining part of this paper, we shall apply them to establish some related properties.

5. Contraction of su(p, q)

For groups other than $\mathrm{U}(p, q)$ (Dobaczewski 1981, 1982, Deenen and Quesne 1984a, b, 1985, Rowe 1984, Hecht and Elliott 1985, Hecht 1985), it has been shown that the unitary operator CS representation of the corresponding Lie algebra is intimately connected with a Dyson boson realisation of the latter (Dyson 1956), from which a Holstein–Primakoff boson realisation can then be derived (Holstein and Primakoff 1940). In addition, for the $\mathrm{Sp}(2d, R)$ positive discrete series irreps, it has been proved that a boson algebra arises in a straightforward way when contracting the Lie algebra $\mathrm{sp}(2d, R)$ in the limit $n \to \infty$, and that such a boson limit is but the leading term in a expansion of the Holstein–Primakoff realisation of $\mathrm{sp}(2d, R)$ into powers of $1/n$ (Rosensteel and Rowe 1981, Rowe and Rosensteel 1982, Deenen and Quesne 1982, Rowe 1984, Castaños and Frank 1985). The purpose of the present section and of the following one is to extend these two properties to $\mathrm{U}(p, q)$. In this section, we shall review the contraction of the $\mathrm{su}(p, q)$ algebra in the large n limit.

Let us define the contracted operators as follows:

$$\begin{aligned} a^{\dagger}_{\alpha\beta} &= \lim_{n\to\infty}(D^{\dagger}_{\alpha,p+\beta}/\sqrt{n}) \qquad & a_{\alpha\beta} &= \lim_{n\to\infty}(D_{\alpha,p+\beta}/\sqrt{n}) \\ \bar{A}_{\alpha\alpha'} &= \lim_{n\to\infty} \bar{E}_{\alpha\alpha'} \qquad & \bar{B}_{\beta\beta'} &= \lim_{n\to\infty} \bar{E}_{p+\beta,p+\beta'} \\ I &= \lim_{n\to\infty}(\mathrm{d}\bar{E}/2npq). \end{aligned} \tag{5.1}$$

By using (2.1), (2.4) and (2.12), we obtain the commutators

$$[a_{\alpha\beta}, a^{\dagger}_{\alpha'\beta'}] = \delta_{\alpha\alpha'}\,\delta_{\beta\beta'} I \tag{5.2a}$$

$$\begin{aligned} &[\bar{A}_{\alpha\alpha'}, \bar{A}_{\alpha''\alpha'''}] = \delta_{\alpha'\alpha''}\bar{A}_{\alpha\alpha'''} - \delta_{\alpha\alpha'''}\bar{A}_{\alpha''\alpha'} \\ &[\bar{B}_{\beta\beta'}, \bar{B}_{\beta''\beta'''}] = \delta_{\beta'\beta''}\bar{B}_{\beta\beta'''} - \delta_{\beta\beta'''}\bar{B}_{\beta''\beta'} \end{aligned} \tag{5.2b}$$

$$\begin{aligned} &[\bar{A}_{\alpha'\alpha''}, a^{\dagger}_{\alpha\beta}] = \delta_{\alpha''\alpha}a^{\dagger}_{\alpha'\beta} - p^{-1}\,\delta_{\alpha'\alpha''}a^{\dagger}_{\alpha\beta} \\ &[\bar{B}_{\beta'\beta''}, a^{\dagger}_{\alpha\beta}] = \delta_{\beta''\beta}a^{\dagger}_{\alpha\beta'} - q^{-1}\,\delta_{\beta'\beta''}a^{\dagger}_{\alpha\beta} \end{aligned} \tag{5.2c}$$

$$\begin{aligned} &[\bar{A}_{\alpha'\alpha''}, a_{\alpha\beta}] = -\delta_{\alpha'\alpha}a_{\alpha''\beta} + p^{-1}\,\delta_{\alpha'\alpha''}a_{\alpha\beta} \\ &[\bar{B}_{\beta'\beta''}, a_{\alpha\beta}] = -\delta_{\beta'\beta}a_{\alpha\beta''} + q^{-1}\,\delta_{\beta'\beta''}a_{\alpha\beta} \end{aligned} \tag{5.2d}$$

all the remaining ones vanishing. Hence the pq pairs of operators $a^{\dagger}_{\alpha\beta}, a_{\alpha\beta}$, $\alpha = 1, \ldots, p$, $\beta = 1, \ldots, q$, are boson creation and annihilation operators, I is the unit operator and $\bar{A}_{\alpha\alpha'}$, $\alpha, \alpha' = 1, \ldots, p$, $\bar{B}_{\beta\beta'}$, $\beta, \beta' = 1, \ldots, q$, respectively, generate $\mathrm{su}(p)$ and $\mathrm{su}(q)$ algebras, with respect to which the $a^{\dagger}_{\alpha\beta}$ behave as vector operators. We conclude that in the large n limit, $\mathrm{su}(p, q)$ contracts to the semi-direct sum of $\mathrm{su}(p) \oplus \mathrm{su}(q)$ and a Heisenberg–Weyl algebra in pq dimensions.

From (2.7) and (2.9), the carrier space $\mathscr{F}_{[k;k']}$ of an irrep $[\boldsymbol{k}; \boldsymbol{k}']$ contracts to the direct product space $\mathscr{B} \times \mathscr{S}$, where $\mathscr{B}$ is the pq boson Fock space built from the operators $a^{\dagger}_{\alpha\beta}$ acting on a vacuum state $|0]$ and $\mathscr{S}$ is the KK'-dimensional carrier space of the unique irrep $[\boldsymbol{k}] \times [\boldsymbol{k}']$ of a $\mathrm{U}(p) \times \mathrm{U}(q)$ intrinsic group. The latter is generated by some operators $\mathring{A}_{\alpha\alpha'}$, $\alpha, \alpha' = 1, \ldots, p$ and $\mathring{B}_{\beta\beta'}$, $\beta, \beta' = 1, \ldots, q$, satisfying the usual $\mathrm{U}(p)$ and $\mathrm{U}(q)$ commutation relations (5.2b) and commuting with one another as well as with the boson operators.

Let us denote the bases of the intrinsic space $\mathscr{S}$ by $|(k), (k')]$, specified by $\mathrm{U}(p)$ and $\mathrm{U}(q)$ Gel'fand patterns (k) and (k'). Provided they are multiplied by $n^{-\frac{1}{2}\Sigma_\beta h_\beta}$, the bases (2.9) of $\mathscr{F}_{[k;k']}$ contract to the bases of $\mathscr{B} \times \mathscr{S}$:

$$\begin{aligned} |\omega[\mathbf{h}][\mathbf{h}'](h)(h')] = B_\omega \sum_{(k)(k')(h^{\backprime})(h^{\backprime})} &\left\langle \begin{matrix} [\mathbf{k}] & [\mathbf{h}^{\backprime}\dot{0}] \\ (k) & (h^{\backprime}) \end{matrix} \middle| \chi; \begin{matrix} [\mathbf{h}] \\ (h) \end{matrix} \right\rangle \\ \times &\left\langle \begin{matrix} [\mathbf{k}'] & [\mathbf{h}^{\backprime}] \\ (k') & (h^{\backprime}) \end{matrix} \middle| \chi'; \begin{matrix} [\mathbf{h}'] \\ (h') \end{matrix} \right\rangle P_{[\mathbf{h}^{\backprime}](h^{\backprime})(h^{\backprime})}(a^{\dagger}_{\alpha\beta})|0]\,|(k), (k')] \end{aligned} \tag{5.3}$$

for which we use a square bracket instead of an angular one. Here $P_{[h^s](h^s)(h'^s)}(a^\dagger_{\alpha\beta})$ is obtained from $P_{[h^s](h^s)(h'^s)}(D^\dagger_{\alpha,p+\beta})$ by substituting $a^\dagger_{\alpha\beta}$ for $D^\dagger_{\alpha,p+\beta}$ and B_ω is defined in such a way that the states (5.3) are normalised to unity.

When su(p, q) is restricted to $\mathscr{F}_{[k;k']}$, the contracted operators $\bar{A}_{\alpha\alpha'}$ and $\bar{B}_{\beta\beta'}$ can be built from the boson operators $a^\dagger_{\alpha\beta}$, $a_{\alpha\beta}$ and the $\mathrm{U}(p)\times\mathrm{U}(q)$ intrinsic group generators $\mathring{A}_{\alpha\alpha'}$, $\mathring{B}_{\beta\beta'}$, as follows:

$$\bar{A}_{\alpha\alpha'}=A_{\alpha\alpha'}-p^{-1}\,\delta_{\alpha\alpha'}\sum_{\alpha''}A_{\alpha''\alpha''}\qquad \bar{B}_{\beta\beta'}=B_{\beta\beta'}-q^{-1}\,\delta_{\beta\beta'}\sum_{\beta''}B_{\beta''\beta''} \tag{5.4}$$

where

$$A_{\alpha\alpha'}=[\boldsymbol{a}^\dagger\tilde{\boldsymbol{a}}+\mathring{\boldsymbol{A}}]_{\alpha\alpha'}\qquad B_{\beta\beta'}=[\tilde{\boldsymbol{a}}^\dagger\boldsymbol{a}+\mathring{\boldsymbol{B}}]_{\beta\beta'}. \tag{5.5}$$

It is straightforward to check that the operators (5.4) and (5.5) satisfy (5.2*b, c, d*) as it should be.

6. Boson realisations of u(p, q)

Let us now come back to finite n values and consider the one-to-one mapping:

$$\mathscr{F}_{[k;k']}\to\mathscr{B}\times\mathscr{S}: |\omega[\boldsymbol{h}][\boldsymbol{h}'](h)(h')\rangle\mapsto|\omega[\boldsymbol{h}][\boldsymbol{h}'](h)(h')]. \tag{6.1}$$

Such a mapping does not preserve the scalar product: contrary to (2.9), the states (5.3) are indeed normalised to unity and orthogonal with respect to $[\boldsymbol{h}^s]$, which now characterises the irreps of the U(p) and U(q) groups generated by $(\boldsymbol{a}^\dagger\tilde{\boldsymbol{a}})_{\alpha\alpha'}$ and $(\tilde{\boldsymbol{a}}^\dagger\boldsymbol{a})_{\beta\beta'}$, respectively.

In the mapping (6.1), the U(p, q) generators are mapped onto some polynomials in the operators $a^\dagger_{\alpha\beta}$, $a_{\alpha\beta}$, $\mathring{A}_{\alpha\alpha'}$ and $\mathring{B}_{\beta\beta'}$. As in the Sp$(2d, R)$ case (Deenen and Quesne 1984a, b, 1985, Rowe 1984), the latter are obtained from the CS representation (4.8) by the replacements

$$u_{\alpha\beta}\to a^\dagger_{\alpha\beta}\qquad \nabla_{\alpha\beta}\to a_{\alpha\beta}\qquad \mathring{\mathscr{E}}^{(p)}_{\alpha\alpha'}\to\mathring{A}_{\alpha\alpha'}\qquad \mathring{\mathscr{E}}^{(q)}_{\beta\beta'}\to\mathring{B}_{\beta\beta'}.$$

The result is

$$\begin{aligned}
(E_{\alpha\alpha'})_{\bar{D}}&=[\boldsymbol{a}^\dagger\tilde{\boldsymbol{a}}+\mathring{\boldsymbol{A}}]_{\alpha\alpha'}\\
(E_{p+\beta,p+\beta'})_{\bar{D}}&=[\tilde{\boldsymbol{a}}^\dagger\boldsymbol{a}+\mathring{\boldsymbol{B}}]_{\beta\beta'}\\
(D^\dagger_{\alpha,p+\beta})_{\bar{D}}&=[\mathring{\boldsymbol{A}}\boldsymbol{a}^\dagger+\boldsymbol{a}^\dagger\tilde{\mathring{\boldsymbol{B}}}+(\boldsymbol{a}^\dagger\tilde{\boldsymbol{a}}-p\boldsymbol{I})\boldsymbol{a}^\dagger]_{\alpha\beta}\\
(D_{\alpha,p+\beta})_{\bar{D}}&=a_{\alpha\beta}.
\end{aligned} \tag{6.2}$$

Since $(D_{\alpha,p+\beta})_{\bar{D}}$ is not the Hermitian conjugate of $(D^\dagger_{\alpha,p+\beta})_{\bar{D}}$ (6.2) provides a Dyson boson realisation of u(p, q). This could have been anticipated from the fact that the mapping (6.1) does not preserve the hermiticity properties of operators.

We can restore the hermiticity properties of operators, namely go from their Dyson realisation $X_{\bar{D}}$ to their Holstein–Primakoff one X_{HP}, by the transformation (Deenen and Quesne 1982, 1984a, b, 1985, Rowe 1984)

$$X_{\mathrm{HP}}=T^{-1/2}X_{\bar{D}}T^{1/2} \tag{6.3}$$

where T is a positive-definite Hermitian operator. By applying (6.3) to the Dyson realisation (6.2) of the generators, we find that the Holstein–Primakoff realisation of the latter is given by

$$\begin{aligned} &(E_{\alpha\alpha'})_{\mathrm{HP}}=[\boldsymbol{a}^{\dagger}\tilde{\boldsymbol{a}}+\mathring{\boldsymbol{A}}]_{\alpha\alpha'} && (E_{p+\beta,p+\beta'})_{\mathrm{HP}}=[\tilde{\boldsymbol{a}}^{\dagger}\boldsymbol{a}+\mathring{\boldsymbol{B}}]_{\beta\beta'} \\ &(D^{\dagger}_{\alpha,p+\beta})_{\mathrm{HP}}=T^{1/2}a^{\dagger}_{\alpha\beta}T^{-1/2} && (D_{\alpha,p+\beta})_{\mathrm{HP}}=T^{-1/2}a_{\alpha\beta}T^{1/2} \end{aligned} \tag{6.4}$$

and that T is a $\mathrm{U}(p)\times\mathrm{U}(q)$ invariant satisfying the matrix relation

$$T\boldsymbol{a}^{\dagger}T^{-1}=\mathring{\boldsymbol{A}}\boldsymbol{a}^{\dagger}+\boldsymbol{a}^{\dagger}\mathring{\tilde{\boldsymbol{B}}}+(\boldsymbol{a}^{\dagger}\tilde{\boldsymbol{a}}-p\boldsymbol{I})\boldsymbol{a}^{\dagger} \tag{6.5}$$

or equivalently

$$T\boldsymbol{a}^{\dagger}T^{-1}=[\Lambda,\boldsymbol{a}^{\dagger}]. \tag{6.6}$$

Here Λ is the $\mathrm{U}(p)\times\mathrm{U}(q)$ invariant defined by

$$\Lambda=\tfrac{1}{2}\operatorname{Tr}[(\boldsymbol{a}^{\dagger}\tilde{\boldsymbol{a}}+\mathring{\boldsymbol{A}})^2+(\tilde{\boldsymbol{a}}^{\dagger}\boldsymbol{a}+\mathring{\boldsymbol{B}})^2-(\boldsymbol{a}^{\dagger}\tilde{\boldsymbol{a}})^2-q\boldsymbol{a}^{\dagger}\tilde{\boldsymbol{a}}] \tag{6.7}$$

and whose eigenvalues are given by

$$\lambda([\boldsymbol{h}^s],[\boldsymbol{h}],[\boldsymbol{h}'])=\tfrac{1}{2}\bigg(\sum_{\alpha=1}^{p}h_\alpha(h_\alpha+n+p-2\alpha+1) + \sum_{\beta=1}^{q}[h'_\beta(h'_\beta+n+q-2\beta+1)-h^s_\beta(h^s_\beta+d-2\beta+1)]+\tfrac{1}{4}dn^2\bigg). \tag{6.8}$$

Determining T is equivalent to finding its matrix elements in the basis (5.3):

$$t_{\omega',\omega}([\boldsymbol{h}],[\boldsymbol{h}'])=[\omega'[\boldsymbol{h}][\boldsymbol{h}'](h)(h')|T|\omega[\boldsymbol{h}][\boldsymbol{h}'](h)(h')]. \tag{6.9}$$

They are diagonal with respect to $[\boldsymbol{h}]$ and $[\boldsymbol{h}']$ and independent of (h) and (h'). From (6.6), we obtain the relation

$$\begin{aligned} &[\omega[\boldsymbol{h}][\boldsymbol{h}']\|Ta^{+}T^{-1}\|\bar{\omega}[\bar{\boldsymbol{h}}][\bar{\boldsymbol{h}}']] \\ &\quad=[\lambda([\boldsymbol{h}^s],[\boldsymbol{h}],[\boldsymbol{h}'])-\lambda([\bar{\boldsymbol{h}}^s],[\bar{\boldsymbol{h}}],[\bar{\boldsymbol{h}}'])][\omega[\boldsymbol{h}][\boldsymbol{h}']\|a^{+}\|\bar{\omega}[\bar{\boldsymbol{h}}][\bar{\boldsymbol{h}}']] \end{aligned} \tag{6.10}$$

where $[\ \|\ \|\ ]$ denotes a $\mathrm{U}(p)\times\mathrm{U}(q)$ reduced matrix element. Whenever the $\mathrm{U}(p)\times\mathrm{U}(q)$ irreps $[\boldsymbol{h}]\times[\boldsymbol{h}']$ and $[\bar{\boldsymbol{h}}]\times[\bar{\boldsymbol{h}}']$ occur with multiplicity one in $[\boldsymbol{k};\boldsymbol{k}']$ (hence the multiplicity labels ω and $\bar{\omega}$ are not needed), (6.10) reduces to

$$t([\boldsymbol{h}],[\boldsymbol{h}'])/t([\bar{\boldsymbol{h}}],[\bar{\boldsymbol{h}}'])=\lambda([\boldsymbol{h}^s],[\boldsymbol{h}],[\boldsymbol{h}'])-\lambda([\bar{\boldsymbol{h}}^s],[\bar{\boldsymbol{h}}],[\bar{\boldsymbol{h}}']) \tag{6.11}$$

where

$$\bar{h}_\alpha=h_\alpha-\delta_{\alpha r} \qquad \bar{h}'_\beta=h'_\beta-\delta_{\beta s} \tag{6.12}$$

for some r, s such that $1\leqslant r\leqslant p$ and $1\leqslant s\leqslant q$.

In particular, for the irreps $[\dot{k};\dot{k}']$, all the $\mathrm{U}(p)\times\mathrm{U}(q)$ irreps are multiplicity free. In such a case, the Holstein–Primakoff realisation of $\mathrm{u}(p,q)$ can be written in a closed analytic form. Since $\mathring{\boldsymbol{A}}=(k+n/2)\boldsymbol{I}$ and $\mathring{\boldsymbol{B}}=(k'+n/2)\boldsymbol{I}$, (6.5) indeed becomes

$$T\boldsymbol{a}^{\dagger}T^{-1}=[\boldsymbol{a}^{\dagger}\tilde{\boldsymbol{a}}+(k+k'+n-p)\boldsymbol{I}]\boldsymbol{a}^{\dagger} \tag{6.13}$$

from which we obtain

$$\begin{aligned} &(E_{\alpha\alpha'})_{\mathrm{HP}}=[\boldsymbol{a}^{\dagger}\tilde{\boldsymbol{a}}+(k+\tfrac{1}{2}n)\boldsymbol{I}]_{\alpha\alpha'} \\ &(E_{p+\beta,p+\beta'})_{\mathrm{HP}}=[\tilde{\boldsymbol{a}}^{\dagger}\boldsymbol{a}+(k'+\tfrac{1}{2}n)\boldsymbol{I}]_{\beta\beta'} \\ &(D^{\dagger}_{\alpha,p+\beta})_{\mathrm{HP}}=\{[\boldsymbol{a}^{\dagger}\tilde{\boldsymbol{a}}+(k+k'+n-p)\boldsymbol{I}]^{1/2}\boldsymbol{a}^{\dagger}\}_{\alpha\beta} \\ &(D_{\alpha,p+\beta})_{\mathrm{HP}}=\{\boldsymbol{a}[\tilde{\boldsymbol{a}}^{\dagger}\boldsymbol{a}+(k+k'+n-q)\boldsymbol{I}]^{1/2}\}_{\alpha\beta}. \end{aligned} \tag{6.14}$$

In (6.14), the square root operators could be written as finite expansions into powers of $\boldsymbol{a}^\dagger\tilde{\boldsymbol{a}}$ or $\tilde{\boldsymbol{a}}^\dagger\boldsymbol{a}$, with coefficients depending upon the invariants $\mathrm{Tr}(\boldsymbol{a}^\dagger\tilde{\boldsymbol{a}})^\lambda$, $\lambda = 0, 1, \ldots, p-1$, or $\mathrm{Tr}(\tilde{\boldsymbol{a}}^\dagger\boldsymbol{a})^\mu$, $\mu = 0, 1, \ldots, q-1$ (Deenen and Quesne 1982).

Whenever $[\boldsymbol{h}]\times[\boldsymbol{h}']$ is not multiplicity free (hence ω is needed), the matrix elements of T can be determined from a recursion relation (Rowe 1984, Castaños *et al* 1985a). By rewriting (6.6) as

$$T = \mathcal{N}^{-1}\sum_{\alpha\beta}[\Lambda, a^\dagger_{\alpha\beta}]Ta_{\alpha\beta} \tag{6.15}$$

where

$$\mathcal{N} = \sum_{\alpha\beta} a^\dagger_{\alpha\beta}a_{\alpha\beta} \tag{6.16}$$

is the boson number operator, and by taking matrix elements of both sides of (6.15), we indeed obtain

$$\begin{aligned} t_{\omega',\omega}([\boldsymbol{h}],[\boldsymbol{h}']) = {} & \left(\sum_\beta h^s_\beta\right)^{-1} \sum_{\bar\omega\bar\omega'[\bar{\boldsymbol{h}}][\bar{\boldsymbol{h}}']} [\lambda([\boldsymbol{h}^{s'}],[\boldsymbol{h}],[\boldsymbol{h}']) - \lambda([\bar{\boldsymbol{h}}^{s'}],[\bar{\boldsymbol{h}}],[\bar{\boldsymbol{h}}'])] \\ & \times[\omega'[\boldsymbol{h}][\boldsymbol{h}']\|a^\dagger\|\bar\omega'[\bar{\boldsymbol{h}}][\bar{\boldsymbol{h}}']][\omega[\boldsymbol{h}][\boldsymbol{h}']\|a^\dagger\|\bar\omega[\bar{\boldsymbol{h}}][\bar{\boldsymbol{h}}']] \\ & \times t_{\bar\omega',\bar\omega}([\bar{\boldsymbol{h}}],[\bar{\boldsymbol{h}}']). \end{aligned} \tag{6.17}$$

In (6.17), the summation over $[\bar{\boldsymbol{h}}]$ and $[\bar{\boldsymbol{h}}']$ runs over those irreps satisfying (6.12) and the reduced matrix element of $a^\dagger$ in the basis (5.3) is given by

$$\begin{aligned} & [\omega[\boldsymbol{h}][\boldsymbol{h}']\|a^\dagger\|\bar\omega[\bar{\boldsymbol{h}}][\bar{\boldsymbol{h}}']] \\ & \quad = U([\boldsymbol{k}],[\bar{\boldsymbol{h}}^s\dot{0}],[\boldsymbol{h}],[1\dot{0}];[\bar{\boldsymbol{h}}]\bar\chi,[\boldsymbol{h}^s\dot{0}]\chi) \\ & \quad\quad \times U([\boldsymbol{k}'],[\bar{\boldsymbol{h}}^s],[\boldsymbol{h}'],[1\dot{0}];[\bar{\boldsymbol{h}}']\bar\chi',[\boldsymbol{h}^s]\chi')[\boldsymbol{h}^s\|a^\dagger\|\bar{\boldsymbol{h}}^s] \end{aligned} \tag{6.18}$$

in terms of SU(p) and SU(q) Racah coefficients and the reduced matrix element of $a^\dagger$ between boson states characterised by $\mathrm{U}(p)\times\mathrm{U}(q)$ irreps $[\boldsymbol{h}^s\dot{0}]\times[\boldsymbol{h}^s]$ and $[\bar{\boldsymbol{h}}^s\dot{0}]\times[\bar{\boldsymbol{h}}^s]$ (Biedenharn and Louck 1968):

$$[\boldsymbol{h}^s\|a^\dagger\|\bar{\boldsymbol{h}}^s] = \left[\left(\prod_{\beta<\beta'}^{q}\frac{(\bar h^s_\beta - \bar h^s_{\beta'} + \beta' - \beta)}{(h^s_\beta - h^s_{\beta'} + \beta' - \beta)}\right)\left(\prod_{\beta=1}^{q}\frac{(h^s_\beta + q - \beta)!}{(\bar h^s_\beta + q - \beta)!}\right)\right]^{1/2}. \tag{6.19}$$

Once the matrix elements of T have been calculated, those of $T^{1/2}$ can be determined by diagonalising the matrix of T. Since in general this requires the solution of high degree algebraic equations, the Holstein–Primakoff realisation of u(p, q) cannot be written in a closed algebraic form (Deenen and Quesne 1985, Castaños *et al* 1985a).

Finally note that in the limit $n\to\infty$, from (6.5) and the properties $\mathring{\boldsymbol{A}}\sim(\frac{1}{2}n)\boldsymbol{I}$, $\mathring{\boldsymbol{B}}\sim(\frac{1}{2}n)\boldsymbol{I}$, we obtain $T\boldsymbol{a}^\dagger T^{-1}\sim n\boldsymbol{a}^\dagger$, so that $T^{1/2}\boldsymbol{a}^\dagger T^{-1/2}\sim n^{1/2}\boldsymbol{a}^\dagger$. Comparison with (5.1) and (6.4) shows that the contracted generators are but the leading term in an expansion of their Holstein–Primakoff realisation into powers of $1/n$.

In the next section, we shall consider some applications of the boson realisations obtained in the present section.

7. Matrix elements of the U(p, q) generators between discrete bases

For practical purposes, it is important to know the matrix elements of the U(p, q) generators in the basis (2.9) corresponding to the chain $\mathrm{U}(p, q)\supset\mathrm{U}(p)\times\mathrm{U}(q)$. In the present section, we shall show that they can be determined from the known matrix elements of boson operators by using the mapping (6.1) and the associated boson realisations of $u(p, q)$. In this way, we shall extend to $U(p, q)$ some results previously demonstrated for $Sp(2d, R)$ (Deenen and Quesne 1984c, 1985, Rowe *et al* 1984, Rowe 1984).

To begin with, we note (Deenen and Quesne 1985) that the operator T, defined in (6.3), is precisely the operator whose matrix elements with respect to the bases (5.3) of $\mathscr{B}\times\mathscr{S}$ reproduce the overlaps of the corresponding bases (2.9) of $\mathscr{F}_{[k;k']}$:

$$t_{\omega',\omega}([\boldsymbol{h}],[\boldsymbol{h}'])=\langle\omega'[\boldsymbol{h}][\boldsymbol{h}'](h)(h')\,|\,\omega[\boldsymbol{h}][\boldsymbol{h}'](h)(h')\rangle. \tag{7.1}$$

The states (2.9) will therefore be normalised to unity provided we replace B_ω in their definition by

$$A_\omega = B_\omega[t_{\omega,\omega}([\boldsymbol{h}],[\boldsymbol{h}'])]^{-1/2}. \tag{7.2}$$

If the boson polynomials $P_{[\boldsymbol{h}^{\backprime}](h^{\backprime})(h^{\backprime})}(a^{\dagger}_{\alpha\beta})$ are normalised in such a way that the highest-weight one is

$$P_{[\boldsymbol{h}^{\backprime}](\max)(\max)}(a^{\dagger}_{\alpha\beta})=\prod_{\beta=1}^{q}(a^{\dagger}_{12\ldots\beta,12\ldots\beta})^{h^{\backprime}_\beta-h^{\backprime}_{\beta+1}} \tag{7.3}$$

where $h^{\backprime}_{q+1}=0$ and $a^{\dagger}_{12\ldots\beta,12\ldots\beta}$ is the determinant of order β obtained from the first β rows and columns of the matrix $\boldsymbol{a}^{\dagger}$, then the normalisation coefficient B_ω of the states (5.3) is given by (Biedenharn and Louck 1968)

$$B_\omega=\left(\prod_{\beta<\beta'}^{q}(h^{\backprime}_\beta-h^{\backprime}_{\beta'}+\beta'-\beta)\right)^{1/2}\left(\prod_{\beta=1}^{q}(h^{\backprime}_\beta+q-\beta)!\right)^{-1/2}. \tag{7.4}$$

The determination of T therefore provides both the overlap of the bases (2.9) and the normalisation coefficient A_ω through (7.1) and (7.2), respectively.

Let us now consider the matrix elements of the $U(p, q)$ generators. Since $E_{\alpha\alpha'}$ and $E_{p+\beta,p+\beta'}$ are $U(p)$ and $U(q)$ generators, respectively, their matrix elements are well known. Moreover, by using the hermiticity property (2.3), the matrix elements of $D_{\alpha,p+\beta}$ can be deduced from those of $D^{\dagger}_{\alpha,p+\beta}$. Hence we are only left with the calculation of the latter. Proceeding as in the case of $Sp(2d, R)$ (Deenen and Quesne 1985), from (6.2) and (6.5) we obtain

$$\begin{aligned}\langle\omega[\boldsymbol{h}][\boldsymbol{h}']\|D^{\dagger}\|\bar\omega[\bar{\boldsymbol{h}}][\bar{\boldsymbol{h}}']\rangle&=[\omega[\boldsymbol{h}][\boldsymbol{h}']\|(D^{\dagger})_{\bar D}\|\bar\omega[\bar{\boldsymbol{h}}][\bar{\boldsymbol{h}}']]\\&=\sum_{\omega'\bar\omega'}t_{\omega,\omega'}([\boldsymbol{h}],[\boldsymbol{h}'])[\omega'[\boldsymbol{h}][\boldsymbol{h}']\|a^{\dagger}\|\bar\omega'[\bar{\boldsymbol{h}}][\bar{\boldsymbol{h}}']]t^{-1}_{\bar\omega',\bar\omega}([\bar{\boldsymbol{h}}],[\bar{\boldsymbol{h}}']).\end{aligned} \tag{7.5}$$

Here $t^{-1}_{\bar\omega',\bar\omega}([\bar{\boldsymbol{h}}],[\bar{\boldsymbol{h}}'])$ denotes a matrix element of T^{-1} and the states

$$|\omega[\boldsymbol{h}][\boldsymbol{h}'](h)(h'))=\sum_{\omega'}|\omega'[\boldsymbol{h}][\boldsymbol{h}'](h)(h')\rangle t^{-1}_{\omega',\omega}([\boldsymbol{h}],[\boldsymbol{h}']) \tag{7.6}$$

for which we use a round bracket instead of an angular one, are dual bases to (2.9). In addition, from (6.4) we get

$$\begin{aligned}\{\omega[\boldsymbol{h}][\boldsymbol{h}']\|D^{\dagger}\|\bar\omega[\bar{\boldsymbol{h}}][\bar{\boldsymbol{h}}']\}&=[\omega[\boldsymbol{h}][\boldsymbol{h}']\|(D^{\dagger})_{\mathrm{HP}}\|\bar\omega[\bar{\boldsymbol{h}}][\bar{\boldsymbol{h}}']]\\&=\sum_{\omega'\bar\omega'}t^{1/2}_{\omega,\omega'}([\boldsymbol{h}],[\boldsymbol{h}'])[\omega'[\boldsymbol{h}][\boldsymbol{h}']\|a^{\dagger}\|\bar\omega'[\bar{\boldsymbol{h}}][\bar{\boldsymbol{h}}']]t^{-1/2}_{\bar\omega',\bar\omega}([\bar{\boldsymbol{h}}],[\bar{\boldsymbol{h}}'])\end{aligned} \tag{7.7}$$

where $t^{1/2}_{\omega,\omega'}([\boldsymbol{h}],[\boldsymbol{h}'])$ and $t^{-1/2}_{\bar\omega',\bar\omega}([\bar{\boldsymbol{h}}],[\bar{\boldsymbol{h}}'])$ are matrix elements of $T^{1/2}$ and $T^{-1/2}$, respectively, and the states

$$|\omega[\boldsymbol{h}][\boldsymbol{h}'](h)(h')\}=\sum_{\omega'}|\omega'[\boldsymbol{h}][\boldsymbol{h}'](h)(h')\rangle t^{-1/2}_{\omega',\omega}([\boldsymbol{h}],[\boldsymbol{h}']) \tag{7.8}$$

for which we use a curly bracket instead of an angular one, are orthonormal bases in $\mathscr{F}_{[k;k']}$. Since the reduced matrix element of $a^\dagger$, appearing in (7.5) and (7.7), is given by (6.18) and (6.19), once again the knowledge of T entirely determines that of the generator matrix elements. In appendix 2, the above general results are illustrated by giving detailed formulae for some special cases.

8. SO(4, 2) coherent states

In this concluding section, we shall indicate how the results of the previous ones can be applied to the SO(4, 2) unitary operator CS by using the isomorphism between so(4, 2) and su(2, 2).

The SO(4, 2) generators are denoted by

$$L_{AB} = -L_{BA} = (L_{AB})^\dagger \qquad A, B = 1, \ldots, 6 \tag{8.1}$$

and satisfy the following commutation relations:

$$[L_{AB}, L_{CD}] = \mathrm{i}(g_{AC}L_{BD} + g_{AD}L_{CB} + g_{BC}L_{DA} + g_{BD}L_{AC}] \tag{8.2}$$

with the metric tensor $g_{AB} = \mathrm{diag}(1, 1, 1, 1, -1, -1)$. An alternative notation (Wolf 1967, Moshinsky and Seligman 1981) is

$$L_a = L_{bc} \qquad A_a = L_{4a} \qquad N_\mu = L_{\mu 5} \qquad K_\mu = L_{\mu 6} \qquad N = L_{56} \tag{8.3}$$

where roman indices take the values 1, 2, 3, with (abc) a cyclic permutation, and greek indices take the values 1, 2, 3, 4. The maximal compact subgroup SO(4) × SO(2), where SO(4) ~ SU(2) × SU(2), is generated by the operators L_a, A_a, $a = 1, 2, 3$, and N which for the hydrogen atom are, respectively, the angular momentum and Runge-Lenz vectors, and the number operator (whose eigenvalue is the principal quantum number). The non-compact generators N_μ and K_μ can be combined into raising and lowering ones, defined by

$$B_\mu^\dagger = N_\mu - \mathrm{i}K_\mu \qquad B_\mu = N_\mu + \mathrm{i}K_\mu \tag{8.4}$$

respectively. The isomorphism between so(4, 2) and su(2, 2) is expressed by the relations

$$\begin{aligned}
&L_+ = L_1 + \mathrm{i}L_2 = E_{12} + E_{34} \qquad && L_0 = L_3 = \tfrac{1}{2}(E_{11} - E_{22} + E_{33} - E_{44})\\
&A_+ = A_1 + \mathrm{i}A_2 = E_{12} - E_{34} \qquad && A_0 = A_3 = \tfrac{1}{2}(E_{11} - E_{22} - E_{33} + E_{44})\\
&B_1^\dagger = D_{13}^\dagger - D_{24}^\dagger \qquad && B_2^\dagger = -\mathrm{i}(D_{13}^\dagger + D_{24}^\dagger)\\
&B_3^\dagger = -D_{14}^\dagger - D_{23}^\dagger \qquad && B_4^\dagger = -\mathrm{i}(D_{14}^\dagger - D_{23}^\dagger)
\end{aligned} \tag{8.5}$$

and those coming from the hermiticity properties $L_- = (L_+)^\dagger$, $A_- = (A_+)^\dagger$ and $B_\mu = (B_\mu^\dagger)^\dagger$.

The SU(2, 2) irreps, i.e. the U(2, 2) irreps $[\boldsymbol{k}; \boldsymbol{k}']$ subject to the equivalence relation (2.13), are (possibly not one-valued) irreps of SO(4, 2), characterised by $\nu_0 s_0 t_0$ or $\nu_0(p_0 q_0)$, where

$$\begin{aligned}
&\nu_0 = \tfrac{1}{2}(k_1 + k_2 + k_1' + k_2' + 2n) \qquad s_0 = \tfrac{1}{2}(k_1 - k_2) \qquad t_0 = \tfrac{1}{2}(k_1' - k_2')\\
&p_0 = s_0 + t_0 \qquad q_0 = s_0 - t_0
\end{aligned} \tag{8.6}$$

respectively, specify the eigenvalue of N corresponding to the LWS and the SU(2)× SU(2) or SO(4) irrep to which it belongs. The discrete bases (2.9) of $\mathscr{F}_{[k;k']}$ are now denoted by $|\omega\nu st\sigma\tau\rangle$ or $|\omega\nu(pq)mm'\rangle$, where ν, s, t, p, q are defined in terms of h_1, h_2, h'_1, h'_2 by relations similar to (8.6), σ, τ, $m=\sigma+\tau$, $m'=\sigma-\tau$ are the eigenvalues of $S_0=\frac{1}{2}(E_{11}-E_{22})$, $T_0=\frac{1}{2}(E_{33}-E_{44})$, L_0 and A_0, respectively, and ω is shorthand for $\nu_s(p_s, 0)$, where $\nu_s=h_1^s+h_2^s+n$, $p_s=h_1^s-h_2^s$. We can go from such bases to states classified according to the chain SO(4) ⊃ SO(3) ⊃ SO(2) by the transformation

$$|\omega\nu(pq)lm\rangle=\sum_{m'}\langle\tfrac{1}{2}(p+q)\tfrac{1}{2}(m+m'), \tfrac{1}{2}(p-q)\tfrac{1}{2}(m-m')|lm\rangle|\omega\nu(pq)mm'\rangle \tag{8.7}$$

where $\langle\ ,\ |\ \rangle$ is an SU(2) Wigner coefficient.

From the previous sections, the unitary operator CS corresponding to an SO(4, 2) irrep $\nu_0(p_0q_0)$ can be written as

$$|\vec{u}, x_1, x_2\rangle=\exp(\vec{u}^*\cdot\vec{B}^\dagger)|x_1, x_2\rangle \tag{8.8}$$

where

$$\vec{u}^*\cdot\vec{B}^\dagger=\sum_\mu u_\mu^* B_\mu^\dagger \tag{8.9}$$

and

$$|x_1, x_2\rangle=\exp(x_1^*L_+ + x_2^*A_+)|\text{LWS}\rangle. \tag{8.10}$$

Here the 4-vector $\vec{u}$ is defined in terms of the 2×2 matrix $\boldsymbol{u}$ of (A1.8) by

$$\begin{aligned}&u_1=\tfrac{1}{2}(u_{11}-u_{22}) \qquad u_2=-\tfrac{1}{2}\mathrm{i}(u_{11}+u_{22}) \qquad u_3=-\tfrac{1}{2}(u_{12}+u_{21})\\ &u_4=-\tfrac{1}{2}\mathrm{i}(u_{12}-u_{21})\end{aligned} \tag{8.11}$$

and satisfies the conditions $\vec{u}\cdot\vec{u}^*<1$ and $1-2\vec{u}\cdot\vec{u}^*+(\vec{u}\cdot\vec{u})(\vec{u}^*\cdot\vec{u}^*)>0$; in addition, x_1 and x_2 are given in terms of the parameters y and z of (A1.8) by

$$x_1=\tfrac{1}{2}(y+z) \qquad x_2=\tfrac{1}{2}(y-z). \tag{8.12}$$

Equation (8.10) is valid whenever $-p_0<q_0<p_0$. It can also be used in the remaining cases provided we set $x_2=x_1$, $x_2=-x_1$ or $x_1=x_2=0$ according as $p_0=q_0>0$, $p_0=-q_0>0$ or $p_0=q_0=0$.

The CS reproducing kernel is

$$\begin{aligned}&\langle\vec{u}', x'_1, x'_2|\vec{u}, x_1, x_2\rangle\\ &\quad=[1-2\vec{u}'\cdot\vec{u}^*+(\vec{u}'\cdot\vec{u}')(\vec{u}^*\cdot\vec{u}^*)]^{-\nu_0-p_0}(\vec{y}\cdot\vec{U})^{p_0+q_0}(\vec{z}\cdot\vec{V})^{p_0-q_0}\end{aligned} \tag{8.13}$$

where the 4-vectors $\vec{y}$, $\vec{z}$, $\vec{U}$ and $\vec{V}$ are defined by

$$\begin{aligned}&y_1=y'+y^* \qquad y_2=-\mathrm{i}(y'-y^*) \qquad y_3=-1+y'y^* \qquad y_4=1+y'y^*\\ &z_1=z'+z^* \qquad z_2=-\mathrm{i}(z'-z^*) \qquad z_3=-1+z'z^* \qquad z_4=1+z'z^*\\ &U_a=-\mathrm{i}\Big(\sum_{bc}\varepsilon_{abc}u'_bu_c^*-u'_au_4^*+u'_4u_a^*\Big) \qquad a=1,2,3\\ &U_4=1-\vec{u}'\cdot\vec{u}^*\\ &V_a=-\mathrm{i}\Big(\sum_{bc}\varepsilon_{abc}u'_bu_c^*+u'_au_4^*-u'_4u_a^*\Big) \qquad a=1,2,3\\ &V_4=1-\vec{u}'\cdot\vec{u}^*\end{aligned} \tag{8.14}$$

and ε_{abc} is the antisymmetric tensor.

For most irreps, the CS satisfy the unity resolution relation

$$\int \mathrm{d}\hat{\sigma}\,(\vec{u}, x_1, x_2)|\vec{u}, x_1, x_2\rangle\langle\vec{u}, x_1, x_2| = I_{\nu_0(p_0,q_0)} \tag{8.15}$$

with the measure

$$\begin{aligned}\mathrm{d}\hat{\sigma}(\vec{u}, x_1, x_2) = {} & 64\pi^{-\alpha}(p+q+1)(p-q+1)(\nu+p-1)(\nu+q-2)(\nu-q-2)(\nu-p-3)\\ & \times[1-2\vec{u}\cdot\vec{u}^*+(\vec{u}\cdot\vec{u})(\vec{u}^*\cdot\vec{u}^*)]^{\kappa}(\vec{y}\cdot\vec{U})^{\lambda}(\vec{z}\cdot\vec{V})^{\mu}\\ & \times \mathrm{d}\vec{u}\,\mathrm{d}\vec{u}^*\,\mathrm{d}x_1\,\mathrm{d}x_1^*\,\mathrm{d}x_2\,\mathrm{d}x_2^*\end{aligned} \tag{8.16}$$

where $\vec{y}$, $\vec{z}$, $\vec{U}$, $\vec{V}$ are defined by (8.14) where we set $\vec{u}'=\vec{u}$, $x_1'=x_1$, $x_2'=x_2$ and

$$\begin{array}{llll}\alpha=6 & \kappa=\nu_0+p_0-2 & \lambda=-(p_0+q_0+2) & \mu=-(p_0-q_0+2)\\ & & & \text{if}\quad -p_0<q_0<p_0\\ \alpha=5 & \kappa=\nu_0+p_0-3 & \lambda=-(p_0+q_0+2) & \mu=0\\ & & & \text{if}\quad p_0=q_0>0\\ \alpha=5 & \kappa=\nu_0+p_0-3 & \lambda=0 & \mu=-(p_0-q_0+2)\\ & & & \text{if}\quad p_0=-q_0>0\\ \alpha=4 & \kappa=\nu_0-4 & \lambda=0 & \mu=0\\ & & & \text{if}\quad p_0=q_0=0.\end{array} \tag{8.17}$$

The existence of the unity resolution relation (8.15) is subject to the condition $\kappa>0$. The latter is not satisfied for the SO(4, 2) irrep containing the bound states of the hydrogen atom, for which $n=\nu_0=1$ and $p_0=q_0=0$. Hence, in this case the CS $|\vec{u}\rangle$ do not form a continuous basis and great caution must be exercised while using them.

For those irreps for which (8.15) is satisfied, the CS representation of the SO(4, 2) generators is given by

$$\begin{aligned}&\hat{\mathscr{L}}_a=-\mathrm{i}\sum_{bc}\varepsilon_{abc}u_b\nabla_c+\mathring{\mathscr{L}}_a \qquad \hat{\mathscr{A}}_a=\mathrm{i}(u_a\nabla_4-u_4\nabla_a)+\mathring{\mathscr{A}}_a\\ &\hat{\mathscr{N}}=\vec{u}\cdot\vec{\nabla}+\nu_0\\ &\hat{\mathscr{B}}^{\dagger}_{\mu}=2u_\mu(\vec{u}\cdot\vec{\nabla}+\nu_0)-(\vec{u}\cdot\vec{u})\nabla_\mu-2\mathrm{i}\sum_{\mu'}u_{\mu'}\mathring{\mathscr{L}}_{\mu'\mu}\\ &\hat{\mathscr{B}}_\mu=\nabla_\mu\end{aligned} \tag{8.18}$$

where $\nabla_\mu=\partial/\partial u_\mu$ and $\mathring{\mathscr{L}}_a=\mathring{\mathscr{L}}_{bc}$, $\mathring{\mathscr{A}}_a=\mathring{\mathscr{L}}_{4a}$ denote the partial differential operators representing L_a and A_a in the SO(4) CS representation corresponding to the states (8.10), i.e.

$$\begin{aligned}&\mathring{\mathscr{L}}_+=\mathring{\mathscr{L}}_1+\mathrm{i}\mathring{\mathscr{L}}_2=-(x_1^2+x_2^2)\partial_1-2x_1x_2\partial_2+2px_1+2qx_2\\ &\mathring{\mathscr{L}}_-=\mathring{\mathscr{L}}_1-\mathrm{i}\mathring{\mathscr{L}}_2=\partial_1 \qquad \mathring{\mathscr{L}}_0=\mathring{\mathscr{L}}_3=x_1\partial_1+x_2\partial_2-p\\ &\mathring{\mathscr{A}}_+=\mathring{\mathscr{A}}_1+\mathrm{i}\mathring{\mathscr{A}}_2=-2x_1x_2\partial_1-(x_1^2+x_2^2)\partial_2+2qx_1+2px_2\\ &\mathring{\mathscr{A}}_-=\mathring{\mathscr{A}}_1-\mathrm{i}\mathring{\mathscr{A}}_2=\partial_2 \qquad \mathring{\mathscr{A}}_0=\mathring{\mathscr{A}}_3=x_2\partial_1+x_1\partial_2-q\end{aligned} \tag{8.19}$$

with $\partial_1=\partial/\partial x_1$ and $\partial_2=\partial/\partial x_2$. Note that for the hydrogen atom bound states, the differential operators (8.18), where $\mathring{\mathscr{L}}_a=\mathring{\mathscr{A}}_a=0$ and $\nu_0=1$, are still useful since they have the same action on the CS as the corresponding SO(4, 2) generators even if they do not form a representation of the latter.

Finally, the reduced matrix elements of the SO(4, 2) non-compact generators can be calculated by the methods of § 7. Let us consider, for instance, the case of an irrep $\nu_0(00)$. From (A2.3), we obtain for the non-vanishing reduced matrix elements between normalised states

$$\begin{aligned}\{\nu+1(p+1,0)\|D^{+}\|\nu(p0)\} &= \frac{1}{2}\left(\frac{(p+1)(\nu+p-\nu_0+4)(\nu+p+\nu_0)}{p+2}\right)^{1/2}\\ \{\nu+1(p-1,0)\|D^{\dagger}\|\nu(p0)\} &= \frac{1}{2}\left(\frac{(p+1)(\nu-p-\nu_0+2)(\nu-p+\nu_0-2)}{p}\right)^{1/2}.\end{aligned} \tag{8.20}$$

From such results, reduced matrix elements in the spherical basis (8.7) can be determined by a recoupling technique. By rewriting $D^{\dagger}_{\alpha,p+\beta}$, $\alpha, \beta = 1, 2$, in the form $D^{\dagger}_{q_1q_2}$, $q_1, q_2 = \pm\frac{1}{2}$, as follows:

$$D^{\dagger}_{13} = D^{\dagger}_{1/2,1/2} \qquad D^{\dagger}_{14} = D^{\dagger}_{1/2,-1/2} \qquad D^{\dagger}_{23} = D^{\dagger}_{-1/2,1/2} \qquad D^{\dagger}_{24} = D^{\dagger}_{-1/2,-1/2} \tag{8.21}$$

and by defining

$$D^{\dagger}_{\kappa q} = \sum_{q_1q_2} \langle \tfrac{1}{2}q_1, \tfrac{1}{2}q_2|\kappa q\rangle D^{\dagger}_{q_1q_2} \qquad \kappa = 0, 1 \tag{8.22}$$

we indeed get

$$\begin{aligned}\{\nu+1(p'0)l'\|D^{\dagger}_{\kappa}\|\nu(p0)l\} &= \{\nu+1(p'0)\|D^{\dagger}\|\nu(p0)\}\\ &\quad\times\langle((\tfrac{1}{2}p,\tfrac{1}{2})\tfrac{1}{2}p', (\tfrac{1}{2}p,\tfrac{1}{2})\tfrac{1}{2}p', l'|(\tfrac{1}{2}p,\tfrac{1}{2}p)l, (\tfrac{1}{2}\tfrac{1}{2})\kappa, l'\rangle\end{aligned} \tag{8.23}$$

where $\langle\ |\ \rangle$ is an SU(2) recoupling coefficient. At last, reduced matrix elements of $B^{\dagger}$ are obtained by noting that from (8.5), (8.21) and (8.22)

$$B^{\dagger}_q = -\sqrt{2}D^{\dagger}_{1q} \qquad q = 1, 0, -1 \qquad B^{\dagger}_4 = -\mathrm{i}\sqrt{2}D^{\dagger}_{00}. \tag{8.24}$$

In the case of the hydrogen atom bound states (for which $(p0) = (\nu-1, 0)$ is omitted), the results are

$$\begin{aligned}\{\nu+1, l+1\|B^{\dagger}\|\nu l\} &= -\left(\frac{(\nu+l+1)(\nu+l+2)(l+1)}{2l+3}\right)^{1/2}\\ \{\nu+1, l-1\|B^{\dagger}\|\nu l\} &= \left(\frac{(\nu-l)(\nu-l+1)l}{2l-1}\right)^{1/2}\\ \{\nu+1, l\|B^{\dagger}_4\|\nu l\} &= -\mathrm{i}[3(\nu-l)(\nu+l+1)]^{1/2}\end{aligned} \tag{8.25}$$

in agreement with other authors (see, e.g., Moshinsky and Seligman 1981).

Appendix 1. Coherent states for the U(2, 1) and U(2, 2) irreducible representations

The purpose of this appendix is to present complete and detailed results for the U(2, 1) and U(2, 2) CS by particularising the general results of § 4.

A1.1. U(2, 1) coherent states

The U(2, 1) irreps are labelled by $[k_1k_2; k']$. In the case where $k_1 = k_2$, we may use the results of § 3 with $p = 2$, $q = 1$. We shall therefore restrict ourselves to the case where $k_1 > k_2$. The CS are then

$$|\boldsymbol{u}, y\rangle = \exp\left(\sum_{\alpha} u^*_{\alpha} D^{\dagger}_{\alpha 3}\right)|y\rangle \tag{A1.1}$$

where $u_\alpha \equiv u_{\alpha 1}$, $\alpha = 1, 2$, and y is a complex variable varying in the whole complex plane and parametrising the U(2) CS

$$|y\rangle = \exp(y^* E_{12})|\text{LWS}\rangle. \tag{A1.2}$$

The CS reproducing kernel is given by

$$\hat{K}(\boldsymbol{u}', y'; \boldsymbol{u}^*, y^*) = \langle \boldsymbol{u}', y' | \boldsymbol{u}, y\rangle = (\det \boldsymbol{U})^{-k_1-k'-n} S^{k_1-k_2} \tag{A1.3}$$

where $\boldsymbol{U}$ is defined in (3.4) and

$$S = U_{11} - U_{21}y' - U_{12}y^* + U_{22}y'y^*. \tag{A1.4}$$

Provided that $k_1 + k' + n > 2$, the states $|\boldsymbol{u}, y\rangle$ satisfy the unity resolution relation (4.6), with the weight function

$$\hat{f}(\boldsymbol{u}, \boldsymbol{u}^*, y, y^*) = \pi^{-3}(k_1 - k_2 + 1)(k_1 + k' + n - 1)(k_2 + k' + n - 2) \times (\det \boldsymbol{U})^{k_1+k'+n-2} S^{-(k_1-k_2+2)} \tag{A1.5}$$

where $\boldsymbol{U}$ and S are obtained from (3.4) and (A1.4) by setting $\boldsymbol{u}' = \boldsymbol{u}$, $y' = y$.

The CS representation of the U(2, 1) generators is obtained from (4.8) after substituting the following operators for $\mathring{\mathscr{E}}^{(p)}_{\alpha\alpha'}$ and $\mathring{\mathscr{E}}^{(q)}_{\beta\beta'}$:

$$\mathring{\mathscr{E}}^{(p)}_{11} = y\partial + k_2 + \tfrac{1}{2}n \qquad \mathring{\mathscr{E}}^{(p)}_{22} = -y\partial + k_1 + \tfrac{1}{2}n$$

$$\mathring{\mathscr{E}}^{(p)}_{12} = y(k_1 - k_2 - y\partial) \qquad \mathring{\mathscr{E}}^{(p)}_{21} = \partial \tag{A1.6}$$

$$\mathring{\mathscr{E}}^{(q)}_{11} = k' + \tfrac{1}{2}n \tag{A1.7}$$

where $\partial = \partial/\partial y$.

A1.2. U(2, 2) coherent states

The U(2, 2) irreps are labelled by $[k_1 k_2; k_1' k_2']$. Whenever $k_1 = k_2$ and $k_1' = k_2'$, we may use the results of § 3 with $p = q = 2$. There are three remaining cases to be considered: (i) $k_1 > k_2$, $k_1' > k_2'$, (ii) $k_1 > k_2$, $k_1' = k_2' = k'$ and (iii) $k_1 = k_2 = k$, $k_1' > k_2'$. In case (i), the CS are defined by

$$|\boldsymbol{u}, y, z\rangle = \exp\left(\sum_{\alpha\beta} u^*_{\alpha\beta} D^\dagger_{\alpha,2+\beta}\right)|y, z\rangle \tag{A1.8}$$

in terms of the four variables $u_{\alpha\beta}$, $\alpha, \beta = 1, 2$, and the two variables y, z, varying in the whole complex plane and parametrising the U(2) × U(2) CS:

$$|y, z\rangle = \exp(y^* E_{12} + z^* E_{34})|\text{LWS}\rangle. \tag{A1.9}$$

To get the CS for cases (ii) and (iii), we only have to set $z = 0$ or $y = 0$ in (A1.8) and (A1.9).

The CS reproducing kernel is

$$\hat{K}(\boldsymbol{u}', y', z'; \boldsymbol{u}^*, y^*, z^*) = \langle \boldsymbol{u}', y', z' | \boldsymbol{u}, y, z\rangle = (\det \boldsymbol{U})^{-k_1-k_1'-n} S^{k_1-k_2} T^{k_1'-k_2'} \tag{A1.10}$$

where S is given in (A1.4), while T is defined by

$$T = V_{11} - V_{12}z' - V_{21}z^* + V_{22}z'z^*. \tag{A1.11}$$

The states $|\boldsymbol{u}, y, z\rangle$ satisfy the unity resolution relation (4.6) with the weight function

$$\hat{f}(\boldsymbol{u}, \boldsymbol{u}^*, y, y^*, z, z^*) = \pi^{-\alpha}(k_1 - k_2 + 1)(k_1' - k_2' + 1)(k_1 + k_1' + n - 1)(k_1 + k_2' + n - 2) \times (k_2 + k_1' + n - 2)(k_2 + k_2' + n - 3)(\det \boldsymbol{U})^{\kappa} S^{\lambda} T^{\mu} \tag{A1.12}$$

provided the condition $\kappa > 0$ is fulfilled. Here U, S and T are obtained from (3.4), (A1.4) and (A1.11) by setting $\boldsymbol{u}' = \boldsymbol{u}$, $y' = y$, $z' = z$, and the exponents α, κ, λ, μ are given by

$$\begin{array}{llll} \alpha = 6 & \kappa = k_1 + k_1' + n - 2 & \lambda = -(k_1 - k_2 + 2) & \\ \mu = -(k_1' - k_2' + 2) & \text{if} & k_1 > k_2 & k_1' > k_2' \\ \alpha = 5 & \kappa = k_1 + k' + n - 3 & \lambda = -(k_1 - k_2 + 2) & \\ \mu = 0 & \text{if} & k_1 > k_2 & k_1' = k_2' = k' \\ \alpha = 5 & \kappa = k + k_1' + n - 3 & \lambda = 0 & \\ \mu = -(k_1' - k_2' + 2) & \text{if} & k_1 = k_2 = k & k_1' > k_2'. \end{array} \tag{A1.13}$$

The CS representation of the U(2, 2) generators is obtained from (4.8) after substituting the operators (A1.6) for $\mathring{\mathscr{E}}^{(p)}_{\alpha\alpha'}$ and the operators

$$\begin{array}{ll} \mathring{\mathscr{E}}^{(q)}_{11} = z\eth + k_2' + \tfrac{1}{2}n & \mathring{\mathscr{E}}^{(q)}_{22} = -z\eth + k_1' + \tfrac{1}{2}n \\ \mathring{\mathscr{E}}^{(q)}_{12} = z(k_1' - k_2' - z\eth) & \mathring{\mathscr{E}}^{(q)}_{21} = \eth \end{array} \tag{A1.14}$$

where $\eth = \partial/\partial z$ for $\mathring{\mathscr{E}}^{(q)}_{\beta\beta'}$.

Appendix 2. Detailed results for the matrix elements of operators between discrete bases

The purpose of this appendix is to present detailed formulae for the matrix elements of the unit operator and of the U(p, q) generators between discrete bases in some special cases. The first three correspond to multiplicity-free states and the fourth to multiplicity-two ones. For the former, the states (2.9), with A_ω substituted for B_ω, coincide with the orthonormal bases (7.8).

A2.1. The case of U(p, q) irreducible representations $[\dot{k}; \dot{k}']$

From (6.11), for the matrix elements of T we obtain the result

$$t([\boldsymbol{h}], [\boldsymbol{h}']) = \prod_{\beta=1}^{q} [(h^s_\beta + k + k' + n - \beta)!/(k + k' + n - \beta)!] \tag{A2.1}$$

where $h^s_\beta = h_\beta - k = h'_\beta - k'$, $\beta = 1, \ldots, q$ and $h_{q+1} = \ldots = h_p = k$. Hence, from (7.2), the normalisation coefficient A is given by

$$A = \left[\left(\prod_{\beta<\beta'}^{q} (h^s_\beta - h^s_{\beta'} + \beta' - \beta) \right) \left(\prod_{\beta=1}^{q} (k + k' + n - \beta)! \right) \times \left(\prod_{\beta=1}^{q} (h^s_\beta + q - \beta)!(h^s_\beta + k + k' + n - \beta)! \right)^{-1} \right]^{1/2}. \tag{A2.2}$$

The non-vanishing reduced matrix elements of $D^\dagger$ are

$$\{[\boldsymbol{h}][\boldsymbol{h}']\|D^\dagger\|[\bar{\boldsymbol{h}}][\bar{\boldsymbol{h}}']\} = \left\{(h_r^s + q - r)(h_r^s + k + k' + n - r) \times \prod_{\beta<\beta'}^{q} [(h_\beta^s - h_{\beta'}^s + \beta' - \beta - \delta_{\beta r} + \delta_{\beta' r})/(h_\beta^s - h_{\beta'}^s + \beta' - \beta)]\right\}^{1/2} \quad \text{(A2.3)}$$

where $\bar{h}_\alpha = h_\alpha - \delta_{\alpha r}$ and $\bar{h}'_\beta = h'_\beta - \delta_{\beta r}$ for some r such that $1 \leqslant r \leqslant q$.

A2.2. The case of U(p, 1) irreducible representations

The U(p, 1) irreps [$\boldsymbol{k}$; k'] only contain multiplicity-free states since [$\boldsymbol{h}^s$] has only one row, whose length is determined by the relation $h^s = h' - k'$, and moreover the reduction of the product of U(p) irreps $[\boldsymbol{k}] \times [\boldsymbol{h}^s]$ into a sum of U(p) irreps [$\boldsymbol{h}$] is multiplicity free. The matrix elements of T and the normalisation coefficient A are given by

$$t([\boldsymbol{h}], [h']) = \prod_{\alpha=1}^{p} [(h_\alpha + k' + n - \alpha)!/(k' + n - \alpha)!] \quad \text{(A2.4)}$$

and

$$A = \left[[(h' - k')!]^{-1}\left(\prod_{\alpha=1}^{p} (k' + n - \alpha)!/(h_\alpha + k' + n - \alpha)!\right)\right]^{1/2} \quad \text{(A2.5)}$$

respectively.

The non-vanishing reduced matrix elements of $D^\dagger$ are

$$\{[\boldsymbol{h}][h']\|D^\dagger\|[\bar{\boldsymbol{h}}][\bar{h}']\} = \left\{(h_r + k' + n - r + 1) \prod_{\alpha=1}^{p} [(h_r - k_\alpha + \alpha - r + 1)/(h_r - h_\alpha + \alpha - r + 1)]\right\}^{1/2} \quad \text{(A2.6)}$$

where $\bar{h}' = h' - 1$ and $\bar{h}_\alpha = h_\alpha - \delta_{\alpha r}$ for some r such that $1 \leqslant r \leqslant p$.

A2.3. The case of U(2, 2) irreducible representations [k_1k_2; $\dot{k}'$] with $k_1 > k_2$

The U(2, 2) irreps [k_1k_2; $\dot{k}'$] only contain multiplicity-free states since [$\boldsymbol{h}^s$] is determined by the relations $h_1^s = h_1' - k'$, $h_2^s = h_2' - k'$ and the reduction of a product of U(2) irreps is multiplicity free. The matrix elements of T are still given by (A2.4) where $p = 2$, while the normalisation coefficient A is

$$A = \left(\frac{(h_1' - h_2' + 1)(k' + n - 1)!(k' + n - 2)!}{(h_1 + k' + n - 1)!(h_2 + k' + n - 2)!(h_1' - k' + 1)!(h_2' - k')!}\right)^{1/2}. \quad \text{(A2.7)}$$

The non-vanishing matrix elements of $D^\dagger$ correspond to $\bar{h}_\alpha = h_\alpha - \delta_{\alpha r}$, $\bar{h}'_\beta = h'_\beta - \delta_{\beta s}$, $r, s = 1, 2$, and become

$$\{[\boldsymbol{h}][\boldsymbol{h}']\|D^{\dagger}\|[\bar{\boldsymbol{h}}][\bar{\boldsymbol{h}}']\} = \begin{cases} \left(\dfrac{(h_1+k'+n-1)(h_1'-k'+1)(h_1-h_2'-k_2+k'+1)(h_1'-h_2+k_2-k')}{(h_1-h_2+1)(h_1'-h_2'+1)}\right)^{1/2} & \text{if} \quad r=s=1 \\ -\left(\dfrac{(h_1+k'+n-1)(h_2'-k')(h_1-h_1'-k_2+k')(h_2-h_2'-k_2+k'+1)}{(h_1-h_2+1)(h_1'-h_2'+1)}\right)^{1/2} & \text{if} \quad r=1, s=2 \\ \left(\dfrac{(h_2+k'+n-2)(h_1'-k'+1)(h_1-h_1'-k_2+k'+1)(h_2-h_2'-k_2+k')}{(h_1-h_2+1)(h_1'-h_2'+1)}\right)^{1/2} & \text{if} \quad r=2, s=1 \\ \left(\dfrac{h_2+k'+n-2)(h_2'-k')(h_1-h_2'-k_2+k'+2)(h_1'-h_2+k_2-k'+1)}{(h_1-h_2+1)(h_1'-h_2'+1)}\right)^{1/2} & \text{if} \quad r=s=2. \end{cases} \quad \text{(A2.8)}$$

A2.4. A multiplicity-two case corresponding to U(2, 2) irreps $[k_1k_2; k_1'k_2']$ with $k_1 > k_2$ and $k_1' > k_2'$

If we consider polynomials of increasing degree in $D^{\dagger}_{\alpha,p+\beta}$ in (2.9), the first non-multiplicity-free case occurs for second-degree polynomials: for $[h_1h_2] = [k_1+1, k_2+1]$ and $[h_1'h_2'] = [k_1'+1, k_2'+1]$, there are indeed two possibilities for the multiplicity label ω corresponding to $[\boldsymbol{h}^s] = [20]$ and $[11]$. By solving the recursion relation (6.17) with the initial value

$$t_{[\dot{0}],[\dot{0}]}([k_1k_2],[k_1'k_2']) = 1 \quad \text{(A2.9)}$$

we successively obtain

$$\begin{aligned} t_{[10],[10]}([k_1+1, k_2],[k_1'+1, k_2']) &= k_1+k_1'+n \\ t_{[10],[10]}([k_1+1, k_2],[k_1', k_2'+1]) &= k_1+k_2'+n-1 \\ t_{[10],[10]}([k_1, k_2+1],[k_1'+1, k_2']) &= k_2+k_1'+n-1 \\ t_{[10],[10]}([k_1, k_2+1],[k_1', k_2'+1]) &= k_2+k_2'+n-2 \end{aligned} \quad \text{(A2.10)}$$

and

$$\begin{aligned} &t_{[20],[20]}([k_1+1, k_2+1],[k_1'+1, k_2'+1]) \\ &\quad = \tfrac{1}{2}[(k_1+k_1'+n)(k_2+k_2'+n-2)+(k_1+k_2'+n-2)(k_2+k_1'+n-2)] \\ &t_{[11],[11]}([k_1+1, k_2+1],[k_1'+1, k_2'+1]) \\ &\quad = \tfrac{1}{2}[(k_1+k_1'+n)(k_2+k_2'+n-2)+(k_1+k_2'+n)(k_2+k_1'+n)] \\ &t_{[20],[11]}([k_1+1, k_2+1],[k_1'+1, k_2'+1]) \\ &\quad = -\tfrac{1}{2}[(k_1-k_2)(k_1-k_2+2)(k_1'-k_2')(k_1'-k_2'+2)]^{1/2}. \end{aligned} \quad \text{(A2.11)}$$

For instance, for $[k_1k_2;\ k_1'k_2'] = [10;\ 10]$, $[h_1h_2] = [h_1'h_2'] = [21]$ and $n = 3$, the overlap matrix (7.1) and the normalisation coefficients A_ω are given by

$$\|t([21],[21])\| = \frac{1}{2}\begin{pmatrix} 9 & -3 \\ -3 & 21 \end{pmatrix} \qquad A_{[20]} = \tfrac{1}{3} \qquad A_{[11]} = 1/\sqrt{21}. \tag{A2.12}$$

The orthonormal bases (7.8) are

$$\begin{aligned} &|[20][21][21](h)(h')\} \\ &\qquad = 0.481\,22|[20][21][21](h)(h')\rangle + 0.041\,95|[11][21][21](h)(h')\rangle \\ &|[11][21][21](h)(h')\} \\ &\qquad = 0.481\,95|[20][21][21](h)(h')\rangle + 0.313\,43|[11][21][21](h)(h')\rangle. \end{aligned} \tag{A2.13}$$

Some reduced matrix elements of $D^\dagger$ involving these states are

$$\begin{aligned} &\langle[20][21][21]\|D^\dagger\|[10][20][20]\rangle = 0 \\ &\langle[11][21][21]\|D^\dagger\|[10][20][20]\rangle = 3/\sqrt{2} \\ &\{[20][21][21]\|D^\dagger\|[10][20][20]\} = 0.198\,97 \\ &\{[11][21][21]\|D^\dagger\|[10][20][20]\} = 1.486\,74. \end{aligned} \tag{A2.14}$$

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